STRENGTH OF MATERIALS

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REVISED AND ENLARGED

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PREFACE TO THE SECOND EDITION

In the preparation of the second edition, no radical changes have been made. The most important *addition* is the method of Area Moments for deriving the equations of the elastic line of beam. This is given along with the usual method of double integration, and the text has been so arranged that either one may be studied and the other omitted. A chapter on Curved Beams and Hooks has been added, and a part of a chapter on Theories of Failure.

At the request of a number of teachers, the problems have been largely rewritten. A few more illustrative examples have been worked out in the text at points where experience has shown that the students have difficulty in the application of the theory. Either Cambria Steel or Carnegie Pocket Companion may be used as a handbook with this edition.

The author is under obligation to Professors E. H. Wood, E. R. Maurer, R. N. Menefee, R. W. Gay, and O. H. Basquin, and to Dean F. E. Ayer and Mr. J. O. Draffin, who have kindly furnished suggestions and constructive criticisms.

J. E. B.

Columbus, Ohio,
March, 1917.

PREFACE TO THE FIRST EDITION

THIS book is intended to give the student a grasp of the physical and mathematical ideas underlying the Mechanics of Materials, together with enough of the experimental facts and simple applications to sustain his interest, fix his theory, and prepare him for the technical subjects as given in works on Machine Design, Reinforced Concrete, or Stresses in Structures.

It is assumed that the reader has completed the Integral Calculus, and has taken a course in Theoretical Mechanics which includes statics and the moment of inertia of plane areas. Chapters XVI and XVII* give a brief discussion of center of gravity and moment of inertia. Students who have not mastered these subjects should study these chapters before taking up Chapter V (preferably before beginning Chapter I).

The problems, which are given with nearly every article, form an essential part of the development of the subject. They were prepared with the twofold object of fixing the theory and enabling the student to discover for himself important facts and applications. The first problems of each set usually require the use of but one new principle,—the one given in the text which immediately precedes; the later problems aim to combine this principle with others previously studied and with the fundamental operations of Mathematics and Mechanics. The constants given in the data or derived from the results of the problems fall within the range of the figures obtained from actual tests of materials. Many of the problems are taken directly from such measurements. Some of them are from tests made by the author or his colleagues at the Ohio State University; others are from bulletins of the University of Illinois Engineering Experiment Station, from "Tests of Metals" at the Watertown Arsenal, and from the Transactions of the American Society of Civil Engineers.

This book is designed for use with "Cambria Steel," to which references are made by title instead of by page, so that they are adapted to any edition of the handbook.

The author acknowledges his indebtedness for suggestions and criticisms to Professors C. T. Morris, E. F. Coddington, Robert Meiklejohn, K. D. Swartzel, and many others of the Faculty of

*Chapters XIX and XX of the Revised Edition.

the College of Engineering; and to Professor Horace Judd of the Department of Mechanical Engineering for the material for several of the half-tones. He also expresses his obligations to the books which have helped to mold his ideas of the subject,—Johnson's "Materials of Construction," Ewing's "Strength of Materials," and especially the text-books which he has used with his classes,—Merriman's "Mechanics of Materials," Heller's "Stresses in Structures," and Goodman's "Mechanics Applied to Engineering."

The symbols used in the mathematical expressions are much the same as in Heller's "Stresses in Structures."

Columbus, Ohio,
November 6, 1911.

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NOTATION

The symbols which are frequently used in this book are:

a = radius of circle; distance of concentrated load from support.
A, A' = area of cross-section.
b = breadth; breadth of rectangular section; base of triangle; distance of concentrated load from support.
B = some special value of b.
c = distance from neutral axis to extreme fiber; distance of center of curvature of circular beam from center of section; distance in figure.
C = distance from center of curvature of trapezoidal curved beam to intersection of sides.
C_1, C_2, C_3 = integration constants.
d = depth; depth of rectangular section; diameter; distance between parallel axes.
D = some special depth; diameter of boiler.
e = eccentricity of a load on a column; distance in figure.
E = modulus of elasticity.
E_c = modulus of elasticity in compression; modulus of elasticity of concrete.
E_s = modulus of elasticity in shear; tension modulus of elasticity of steel reinforcement.
E_t = modulus of elasticity in tension.
E_v = modulus of volume elasticity.
E_w = working modulus of elasticity.
h = height; distance from vertex to base of triangle.
hp = horsepower.
H = product of inertia.
I = moment of inertia.
I_m = maximum moment of inertia of a beam of variable section.
I_x = moment of inertia with respect to the X axis.
I_y = moment of inertia with respect to the Y axis.
I_o = moment of inertia with respect to an axis through the center of gravity.
j = ratio of moment arm to total depth of a reinforced concrete beam.
J = polar moment of inertia.
k = a constant coefficient; radius of gyration (in Chapter XX); a ratio less than unity.
l = length; length of beam between supports; length of column between points of inflection.
L = length; total length of column.
m = mass of particle; slope of tangent at support; a ratio.
M = moment; mass.
M_o = moment at origin of coördinates.
M_a, M_b, M_c = moment over three consecutive supports.

M_1, M_2, M_3, etc. = moment over first, second, third, etc., supports.

n = ratio, number of turns in a helical spring.

N = normal force at surface; number of revolutions per minute

p = pitch of rivets; slope of tangent; ratio of steel area to concrete area.

P = concentrated load or force.

q = coefficient in Rankine's formula.

Q = concentrated load or force.

r = distance from origin; radius of gyration (in column formulas); radius.

R = reaction at support; resultant force; radius; radius of coil.

R_1 = reaction at left support; radius of inside surface of curved beam or hook.

R_2 = reaction at second support; radius of outside surface of curved beam or hook.

R_o = radius of neutral surface of curved beam or hook

s = unit stress.

s_t, s_s, s_c = unit tensile, shearing, and compressive stress.

s_u = ultimate unit stress.

s_w = allowable unit stress.

s' = unit stress resulting from combined shear and tension or compression.

S = unit stress in extreme fibers.

S_1 = unit stress at concave surface of curved beam.

S_2 = unit stress at convex surface of curved beam.

S_s = unit shearing stress at surface of shaft.

t = thickness.

T = torque; tension.

U = work.

U_p = modulus of resilience.

v = distance from neutral axis.

V = total vertical shear.

V_{ab} = total shear near support A in span joining A to B.

w = distributed load per unit of length.

W = total load uniformly distributed.

$\bar{x}$, $\bar{y}$, $\bar{z}$ = coördinates of center of gravity.

y = deflection in a beam or column.

y_{ab} = deflection at B due to a load at A.

y_{max} = maximum deflection in a beam or column.

δ = unit deformation.

δ_s = unit shearing deformation

μ = coefficient of friction.

σ = Poisson's ratio.

ρ = density; radius of curvature.

$\alpha, \beta, \theta, \phi$ = angles in figure.

STRENGTH OF MATERIALS

CHAPTER I

STRESSES

1. Strength of Materials.—That branch of Mechanics which treats of the changes in form and dimensions of elastic solids and the forces which cause these changes is called *The Mechanics of Materials.* When the physical constants and the results of experimental tests upon the materials of construction are included with the theoretical discussion of the ideal elastic solid, the entire subject is called *The Strength of Materials* or *The Resistance of Materials.*

2. Tension.—Support one end of a band of soft rubber, and attach a small hook to the other end, as shown in Fig. 1. Now apply a small weight to the hook. The rubber band is stretched; its length is increased by an amount a, while its cross-section is diminished. Add a second weight. If the second weight is equal to the first one, the elongation b, which it causes, is equal to that caused by the first weight. Remove the weights, and the rubber band returns to its original length and cross-section.

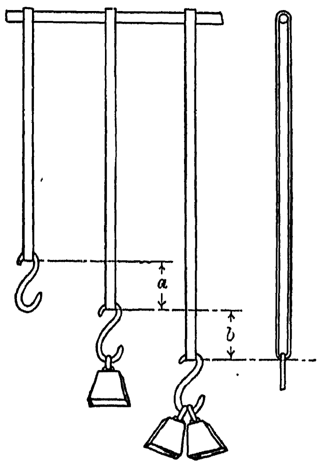


FIG. 1.—Rubber bands in tension.

If steel, iron, wood, concrete, stone, or other solid material is used instead of rubber, the results are similar. There is this apparent difference: while the rubber may be stretched to twice or three times its original length and still return to its original size and shape after the load is removed, one of the other materials may be stretched only a very small amount (usually less than 0.002 of its length), without receiving a permanent change in its dimensions. Again, the force required to produce a relatively small increase in the length of a rod of wood or steel, for instance,

is many times greater than that necessary to *double* the length of a soft rubber band of equal cross-section. These differences between the behavior of soft rubber and other solid materials are differences of degree and not of kind. Essentially they are alike.

The rubber bands shown in Fig. 1 are subjected to the action of two forces: the force of the weights pulling downward, and the reaction of the support pulling upward. The bands are in *tension*. A body is said to be in tension when it is subjected to two sets of forces whose resultants are in the same straight line, opposite in direction, and directed *away* from each other.

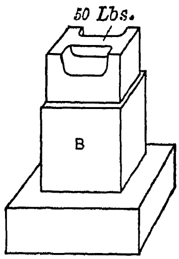


FIG. 2.—Compression.

3. Compression.—When a body is subjected to two sets of forces whose resultants are in the same straight line, opposite in direction, and directed *toward* each other, it is said to be in *compression*. In Fig. 2, the block B is in compression under the action of the 50 pounds pushing down and the reaction of the support pushing up. The effect of compression upon a body is to shorten it in the line of the forces and increase its dimensions in the plane perpendicular to this line.

Tension and compression may be represented as in Fig. 3, where the arrows represent the forces, and the small rectangles represent the bodies, or portions of a body, upon which the forces act. The rectangles are often omitted; a pair of arrows with their heads together indicate compression, and a pair with their heads in the opposite sense indicate tension.

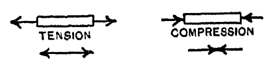


FIG. 3.

4. Stress; Total Stress.—The force exerted by one body upon another at their surface of contact is called the *stress* between the bodies or the *total stress* between the bodies. If a single body be regarded as cut by an imaginary surface, the force exerted across this surface by either portion of the body upon the other portion is the total *internal stress* in the body at the section. In the case of an internal stress, if the forces are such that the portions of the body are pushed together at the imaginary surface, the stress is *compressive*. If the forces tend to pull the portions apart, the stress is *tensile*. Compressive stress at the surface of contact of two separate bodies is called *bearing stress*.

All parts of the bar AB, Fig. 4, are under tensile stress. The total tensile stress at any section CD is the load L and the weight of the hook and of that portion of the bar below the section.

All parts of the block in Fig. 5 are in compression. The total compressive stress at any section JK is 10 pounds plus the weight of the portion of the block above the section; or, since action and

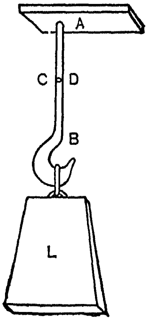


FIG. 4.—Tensile stress.

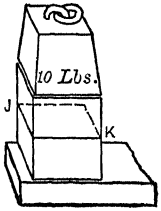


FIG. 5.—Compressive stress.

reaction are equal, it is the upward reaction at the base minus the weight of the portion below JK.

5. Unit Stress; Intensity of Stress.—The *unit stress* at any surface is the total stress at the surface divided by its area. Unit stress is frequently called *intensity* of *stress*. In American engineering practice, unit stresses are usually expressed in pounds per square inch. Compressive stresses in masonry are sometimes given in tons per square foot; the bearing pressure of masonry upon soils is always so expressed. English engineers frequently use long tons per square inch to express the intensity of stress in steel and similar solids. Continental engineers,* of course, use kilograms per square centimeter. Physicists, the world over, prefer dynes per square centimeter. In the case of tensile or compressive stresses, the surface considered is a plane section perpendicular to the direction of the forces, unless otherwise stated.

Pounds per square inch are frequently written lb./in.^{2}

Problems

1. The rod AB, Fig. 4, is circular and 2 inches in diameter. If the load L is 16,000 pounds, and the weight of the hook and the lower part of the rod are neglected, what is the unit stress at any section?

Ans. 5,093 pounds per square inch.

* They sometimes express stress in atmospheres. One atmosphere equals 14.7 lb./in.^{2} equals 1.033 kg./cm.^{2}

2. If in Fig. 4 the diameter of the rod is 3 inches, what must be the load L to produce an intensity of stress of 6,000 pounds per square inch?

3. A pier 16 inches square carries a load of 9,600 pounds. Find the unit compressive stress in the pier in pounds per square inch and the unit bearing stress on the soil beneath it in tons per square foot.

Ans. Bearing stress 2.7 tons per square foot.

4. A short piece of 6-inch wrought-iron water pipe, standing on end, supports a load of 47,040 pounds. What is the unit compressive stress? (See handbook for dimensions.) *Ans.* 8,400 pounds per square inch.

5. One inch equals 2.540 centimeters and 1 pound equals 453.6 grams. Find the value of 1 kilogram per square centimeter in pounds per square inch and compare the result with the handbook.

6. Working Stress; Allowable Unit Stress.—*Working stresses* are the unit stresses to which the materials of a structure or machine are subjected. The *allowable unit stress* for a given material is the maximum working stress which, in the judgment of some engineer or other authority, should be applied to that material. As examples of allowable unit stresses, the building laws of New York City and the American Railway Engineering and Maintenance of Way Association recommend 16,000 pounds per square inch as the allowable unit tensile stress in structural steel. The United States Department of Agriculture gives 1,000 pounds per square inch as the allowable compressive stress, parallel to the grain, in long-leaf yellow pine, and 215 pounds per square inch across the grain.

The following should be memorized:

A steel bar 1 foot in length and 1 square inch in cross-section weighs 3.4 pounds. ie 12 cu. inches = 3.4 lbs.

One cubic foot of water weighs approximately 62.5 pounds.

TABLE I.—AVERAGE ALLOWABLE UNIT STRESS

Material	Pounds per square inch	
	Tension	Compression
Structural steel	16,000	16,000
Wrought iron	12,000	12,000
Cast iron	3,000	15,000
Long-leaf yellow pine	......	1,200 with the grain
Long-leaf yellow pine	......	250 across grain
White oak	......	1,000 with the grain
White oak	......	400 across grain
Portland cement concrete....1 : 2 : 4	......	450
Portland cement concrete....1 : 3 : 6	......	300

Problems

Use data of above table unless otherwise specified.

1. Find the total allowable load in compression, parallel to the grain, which may be applied to a 6-inch by 6-inch short block of long-leaf yellow pine. *Ans.* 43,200 pounds.

2. What is the breadth of a structural steel eye-bar, 1 inch in thickness, which exerts a pull of 96,000 pounds?

3. What should be the dimensions of a cubical block of white oak which supports a load of 64,000 pounds? (Two cases.)

4. What is the allowable pull on a $\frac{3}{4}$-inch wrought iron bolt? (See handbook for dimensions.) *Ans.* 3,624 pounds.

5. A 6-inch by 6-inch short white oak post stands on a cast-iron base-plate which is supported by a pier of 1 : 2 : 4 concrete. If the post is subjected to its allowable safe load, what is the minimum area of the pier? *Ans.* 80 square inches.

6. If the pier in Problem 5 is 12 inches square at the top, 9 feet high, and enlarges to form the frustum of a pyramid, what must be the area of the lower base if the allowable bearing stress on the soil is 2 tons per square foot and the weight of the pier is neglected? Solve also, taking into account the weight of the pier. (Cubic foot of concrete weighs 150 pounds.)

7. A tank weighing 150 pounds and holding 12 cubic feet of water is supported by two $\frac{1}{2}$-inch wrought-iron bolts with ends threaded. Is the construction safe?

7. Deformations; Unit Deformation. The changes in dimensions which occur when forces are applied to a body are called deformations. In Fig. 1, the increase of length, a, which takes place when the first load is applied is the deformation due to that load, the increase b is the deformation due to the second load, and $a + b$ is the deformation due to the two loads. The deformation produced by a *tensile* force or *pull* is an *elongation;* that caused by a *compressive* force or *push* is a *compression.* A deformation which remains after the force is removed is called a *set.*

Unit deformation in a body is the deformation per unit length. In a bar of uniform cross-section it is calculated by dividing the total deformation in a given length by the original length. The length a in Fig. 1 divided by the original length of the band is the unit elongation due to the first load. It is frequently convenient to consider unit deformation as the ratio of the deformation to the original length. It is then called the *relative deformation.*

In algebraic equations many authors represent unit deformation by the letter δ (pronounced delta).

Deformation is frequently called *strain.* The word, *strain,* formerly was used as a synonym for *stress,* and it is still sometimes heard in that sense; but the general practice of technical litera-

ture, at present, is to use *strain* to mean *deformation*. When employed in this book it will always have this meaning.

Problems

1. A bar is subjected to a tensile stress and a portion of it, originally 8 inches long, is stretched 0.0052 inch. Find the unit elongation.
Ans. 0.00065 ~~inch~~.

2. A wooden post under compression is shortened 0.144 inch in a length of 16 feet. Find the unit compression. *Ans.* 0.00075.

3. The coefficient of expansion of iron is 0.000012 for 1°C. What is the unit elongation and the total elongation in an iron rod 20 feet long when the temperature changes from 50°F. to 20°C.?

4. In the tension test of a bar of cast iron, it was found that a pull of 7,000 pounds per square inch produced an elongation of 0.0044 inch in a length of 8 inches. What was the relative elongation? *Ans.* 0.00055.

8. Elastic Limit.—When a stress is applied to a solid body and then removed, the body returns to its original size and shape, provided the stress has not exceeded a certain limit. If the stress has gone beyond this limit, the body does not return entirely to its original dimensions, but retains some permanent deformation or set. The *unit stress* at this limit is called the *elastic limit* of the material. A wrought-iron rod is stretched about 0.006 inch in a length of 8 inches by a pull of 20,000 pounds per square inch. When the load is removed it returns to its original length. The unit stress of 20,000 pounds per square inch is below the elastic limit of wrought iron. If the load is increased to 30,000 pounds per square inch, the elongation in 8 inches becomes, perhaps, 0.075 inch. When this load is removed the rod shortens only 0.009 inch while the remaining 0.066 inch persists as a permanent set. The elastic limit is below 30,000 pounds per square inch.

The elastic limit cannot be determined exactly. A test piece may appear to have no permanent deformation when measured with the usual apparatus and still show some set when more delicate instruments are employed. Time also is a factor. If a load is applied for a considerable period, it causes somewhat greater deformation and relatively considerably greater set than if the time of application is short. Some materials, such as steel, after having been subjected to comparatively large unit stress, shows some deformation shortly after the load is removed, which may partly or entirely vanish after some little interval.

9. Modulus of Elasticity.—For all stresses below the elastic limit the unit stress bears a constant ratio to the unit deformation. The quotient obtained by dividing unit stress by the accompanying unit deformation is called the *modulus* of *elasticity* of the material, or *Young's* modulus. In algebraic formulas, modulus of elasticity is represented by the letter E. Writing the above definition algebraically.

$$E = \frac{s}{\delta}, \qquad \text{* Formula I.}$$

where E is the modulus of elasticity,
s is the unit stress,
δ is the unit deformation,

Problems

1. A steel rod of 1 square inch cross-section is tested in tension. It is found that a pull of 17,700 pounds stretches 8 inches of the rod 0.0048 inch. Find the unit elongation and the modulus of elasticity.

Ans. Modulus of elasticity equals 29,500,000 pounds per square inch.

2. A wooden block 4 inches square and 16 inches long is tested in compression. It is found that a total load of 8,000 pounds shortens 10 inches of the block 0.0040 inch. Find the modulus of elasticity of this wood.

Ans. 1,250,000 pounds per square inch.

3. A bridge post made of two 12-inch channels, each weighing 25 pounds per foot, was shortened 0.008 inch in a length of 40 inches by the weight of a moving train. If $E = 29{,}000{,}000$ pounds per square inch for this steel, find the additional load due to the train which this post carried.

Ans. 85,260 pounds.

4. In a tension test of cast iron at the Watertown Arsenal, an increase of unit stress from 1,000 pounds per square inch to 6,000 pounds per square inch produced an increase in length of 0.0034 inch in a gage length of 10 inches. Find E for this cast iron.

5. A wrought-iron column, tested at Watertown Arsenal, was 11.31 square inches in cross-section. When the load was changed from 5,000 pounds to 100,000 the column shortened 0.0610 inch in a gage length of 200 inches. Find E for this wrought iron.

The following values of the modulus of elasticity for some common materials in direct tension or compression should be memorized.

* Important formulas, which should be memorized, are designated by Roman numerals in this book.

TABLE II.—MODULUS OF ELASTICITY

Material	Modulus, in pounds per square inch
Structural steel	29,000,000
Hard steel	30,000,000
Wrought iron	27,00,0000
Cast iron	15,000,000
Timber (parallel to the grain)	1,000,000 to 2,000,000
Portland cement concrete	2,000,000 to 4,000,000

Problems

6. A bar of structural steel, 1 inch by 2 inches, is stretched 0.0048 inch in a length of 8 inches. What is the load applied? *Ans.* 34,800 pounds.

7. A cast-iron rod is subjected to its allowable tensile stress. What is the elongation in a length of 50 inches?

8. The temperature coefficient of steel is 0.0000067 per degree Fahrenheit. How much stress is developed in a rod of structural steel when the temperature changes from 80°F. to 20°F. and the rod is not allowed to contract?

9. A foreign handbook gives the modulus of elasticity in kilograms per square millimeter as follows:

Steel	20,400
Copper (drawn)	12,400
Brass	10,800
Aluminum (drawn)	7,500

Reduce these to pounds per square inch.

10. A cast-iron bar 2 inches wide and 1 inch thick is placed between two structural-steel bars each 2 inches wide and ⅝ inch thick. What total pull will stretch the combined bar 0.0018 inch in a length of 8 inches, and what will be the unit stress in each material? Does either unit stress exceed the allowable value? *Ans.* Total pull, 23,062 pounds.

11. A structural-steel bar 6 inches wide and 1 inch thick is placed between two wrought-iron bars each 4 inches wide and ¾ inch thick, and a total pull of 125,100 pounds is applied to the combination. What is the unit tensile stress in each material?

12. A 1-inch round steel rod passes through a brass tube 1 inch inside diameter and 2 inches outside diameter. A nut on the steel rod is turned until the tensile stress in it is 6,000 pounds per square inch. Find the unit compressive stress in the brass tube. How much additional stress is developed in each if the temperature is raised 40°C., and the coefficient of expansion of the steel is 0.000012 and that of the brass 0.000018? Use 29,000,000 as the modulus of the steel and take the modulus of the brass from the result of Problem 9.

10. Physical Meaning of E.—Formula I of Article 9 may be written

$$\delta = \frac{s}{E}.$$

If s be made equal to unity, δ becomes equal to $\frac{1}{E}$. With the common engineering units, the reciprocal of E is the unit deformation produced by a unit load of 1 pound per square inch. For steel having a modulus of 30,000,000 pounds per square inch, a unit stress of 1 pound per square inch is developed when the deformation is one thirty-millionth of the original length.

Examples

Solve without writing

1. If wood having a modulus of 1,500,000 pounds per square inch is subjected to a tensile stress of 600 pounds per square inch, what is its elongation per inch of length? What is the total elongation in a length of 5 feet?

2. A 2-inch by 2-inch wooden block is subjected to a compressive load of 2,000 pounds. If the modulus of elasticity parallel to the fibers is 1,500,000 pounds per square inch, what is the unit compression, and what is the total compression in a length of 24 inches.

3. If steel has a modulus of 30,000,000 pounds per square inch, what is the unit elongation due to a stress of 15,000 pounds per square inch? What will be the elongation in a steel tape 200 feet long due to this stress?

4. What is the unit elongation in cast iron when the unit tensile stress is at its allowable value?

Formula I may also be written

$$s = E\delta,$$

which defines E as the coefficient which multiplied into the unit deformation gives the unit stress. It helps to fix our ideas if we consider the case where the unit deformation is 0.001. We may then define the modulus as 1,000 times the unit stress which produces a unit deformation of 0.001 of the original length.

Examples

Solve without writing

5. If the modulus of steel is 30,000,000, what is the unit stress when the unit deformation is 0.001? If the unit deformation is 0.0005, what is the unit stress? If a steel rod 40 inches long is stretched 0.008 inch, what is the unit stress? What total pull will stretch a bar 2 inches square 0.024 inch in a length of 5 feet?

6. If the modulus of white oak is 1,500,000, what is the unit stress which produces a unit elongation of 0.001? Is this more or less than the allowable unit compressive stress?

If δ be made equal to unity in Formula I, s becomes equal to E. From this relation the modulus of elasticity in tension is sometimes defined as the unit stress which would double the length of a rod of uniform cross-section, if such doubling were possible without breaking the rod or exceeding the elastic limit.

11. Work and Resilience.—When a force acts on a body and motion takes place in the direction of the force, the force is said to do *work*. The distance which the point of application moves is called the *displacement*. The work done by a constant force is the product of the force multiplied by the displacement. If P represents the constant force and x represents the displacement of its point of application, the work is the product Px, provided the force and displacement are in the same direction. If the force is in pounds and the displacement is in feet, the work is expressed in *foot-pounds*. If the force is not constant the work is the product of the *average force* multiplied by the displacement. When an elastic body is deformed, the force varies directly as the displacement (provided the elastic limit is not exceeded) and the average force is half the sum of the initial and final forces.

Suppose a given spring is stretched 3 inches by a load of 24 pounds, and it is required to find the work done on the spring by the load. The average pull is 12 pounds and the total work is 12 pounds multiplied by 3 inches, or 36 inch-pounds. Now suppose an additional 16 pounds be added, producing an additional elongation of 2 inches and making the total elongation 5 inches in all. To find the work done when the additional 16 pounds is applied we take the mean of 24 pounds and 40 pounds and multiply by 2 inches, from which we get 64 inch-pounds. If the entire 40 pounds be applied at one time,

$$U = \frac{0 + 40}{2} \times 5 = 100 \text{ inch-pounds.}$$

Problems

1. A given spring is stretched 1 inch by a load of 12 pounds. How much will a load of 48 pounds stretch it and what is the work done on the spring in foot-pounds? *Ans.* 8 foot-pounds.

2. If the spring in Problem 1 carries an initial load of 48 pounds and an additional load of 36 pounds be applied, producing an additional elongation of 3 inches, find the work done in stretching the spring these 3 inches. *Ans.* 16.5 foot-pounds.

3. If to the spring of Problem 1 a load of 84 pounds is applied when there is no initial load, what is the total work? Compare the result with the answers of Problems 1 and 2.

4. Find the work done in stretching the spring of Problem 1 a distance of 10 inches with no initial elongation. Also, find the work done in stretching it 10 inches from an initial elongation of 2 inches.

5. A load of 7,200 pounds is applied to a steel rod having no initial load and stretches it 0.02 inch. Find the work in foot-pounds.

Ans. 6 foot-pounds.

6. A pull of 60,000 pounds is applied to a steel rod of 5 square inches cross-section. If the modulus of the steel is 30,000,000 pounds per square inch, what is the work done in a length of 20 feet? *Ans.* 240 foot-pounds.

7. What would be the work in Problem 6 if the load were applied to a rod of 3 square inches cross-section and 20 feet in length?

8. The allowable compressive stress is applied to a cast-iron bar 2 inches square and 10 inches long. Find the work in foot-pounds.

Ans. 25 foot-pounds.

The work done in deforming an elastic body is stored up as elastic energy, which may be recovered as mechanical work when the load is removed. This elastic energy is called the *resilience* of the material. If the unit stress does not exceed the elastic limit, practically all the work which is put into it is recovered. If it goes beyond the elastic limit, part of the work is lost.

12. Modulus of Resilience.—The work expended in deforming unit volume of any material to the elastic limit is called the *modulus of resilience* of the material. It is the *elastic potential energy* of unit volume when stressed to the elastic limit. The modulus of resilience is a *measure* of the amount of energy which may be stored in a given material and recovered as mechanical work without loss.

If we consider a cubic inch of material subjected to unit stress s, the deformation is $\frac{s}{E}$ and the average force is $\frac{s}{2}$; the total work

$$U_p = \frac{s}{E} \times \frac{s}{2} = \frac{s^2}{2E}. \qquad \text{Formula II.}$$

This expression $\left(\text{energy in unit volume} = \frac{s^2}{2E}\right)$ gives the energy for any value of s below the elastic limit. When s is the unit stress at the elastic limit, the expression is the modulus of resilience. When s and E, are given in pounds per square inch, Formula II gives the energy in *inch-pounds* per *cubic inch.*

The total elastic energy in a body, all parts of which are subjected to a unit stress s, is obtained by multiplying the total volume of the body by the energy per unit volume, and is independent of the form of body.

Problems

1. Find the modulus of resilience of structural steel having a modulus of elasticity of 29,000,000 pounds per square inch and an elastic limit of 32,000 pounds per square inch. *Ans.* 17.6 inch-pounds per cubic inch.

2. Find the modulus of resilience of spring steel for which E equals 30,000,000 pounds per square inch and the elastic limit is 90,000 pounds per square inch. *Ans.* 135 inch-pounds.

3. A 2-inch round steel rod is subjected to a pull of 80,000 pounds which produces an elongation of 0.0105 inch in a 12-inch length. What is the total work and the work per cubic inch?

4. What is the modulus of resilience of timber for which the modulus of elasticity is 1,500,000 pounds per square inch and the elastic limit is 3,000 pounds per square inch?

5. How high can the energy stored in a cubic inch of the steel of Problem 2 lift its own weight? *Ans.* 39.7 feet.

6. How high can the energy stored in the timber of Problem 4 lift its weight, if a cubic foot of this material weighs 36 pounds?

In calculating the work of resilience, we used the *average force* multiplied by the deformation. We may obtain the same results by means of the Calculus.

Let x represent the total elongation of a rod of length l and unit cross-section; and let dx represent an infinitesimal increment of this elongation. When the elongation is x the unit elongation is $\frac{x}{l}$ and the unit stress is $\frac{Ex}{l}$.

The work done in causing an elongation dx in the rod of unit cross-section is the product of this unit stress multiplied by dx.

$$\text{Increment of work} = \frac{Ex}{l}\,dx. \tag{1}$$

$$\text{Total work} = \int \frac{Ex}{l}\,dx = \frac{E}{2\,l}\Big[x^2\Big]_{x_1}^{x_2} = \frac{E}{2\,l}(x_2^2 - x_1^2), \tag{2}$$

where x_1 and x_2 are the initial and final elongations respectively. Substituting for x_1 and x_2 their values in terms of the stress, we get:

$$\text{Total work} = l\left(\frac{s_2^2 - s_1^2}{2\,E}\right) = \left(\frac{s_2^2 - s_1^2}{2\,E}\right) \times \text{volume}. \tag{3}$$

If the initial stress is zero, equation (3) becomes Formula II.

Problems

7. Derive equation (3) by means of average force, without integrating.

8. Find the work done on 100 cubic inches of cast iron in compression when the unit stress changes from 0 to 5,000 pounds per square inch, and when it changes from 5,000 to 10,000 pounds per square inch.

13. Poisson's Ratio. When a body is subjected to a tensile stress it is elongated, the amount of elongation, provided the unit stress does not exceed the elastic limit, being proportional to the stress. At the same time its diameter is diminished. The ratio of this relative decrease in diameter to the unit increase in length is called Poisson's ratio. The value of this ratio varies with the material, but it is usually in the neighborhood of $\frac{1}{4}$. It is about 0.27 for steel. If a steel rod is elongated 0.001 of its length, its diameter is diminished about 0.00027 of its original value. The same relation holds in compression.

Poisson's ratio will be represented in this book by the Greek letter σ.*

Problems

1. Taking Poisson's ratio as 0.27 and the modulus of elasticity as 29,000,000 pounds per square inch, find the decrease in width of a 1-inch by 5-inch steel bar due to a pull of 100,000 pounds? *Ans.* 0.00093 inch.

2. In Problem 1, if the unit stress is proportional to the unit deformation, what is the transverse unit compressive stress?

Ans. 5,400 pounds per square inch.

3. Poisson's ratio for copper is $\frac{1}{3}$ and E is about 16,000,000 pounds per square inch. How much is the diameter of a 2-inch round copper rod increased when subjected to a compressive load of 25,000 pounds?

Ans. 0.00033 inch.

4. A steel plate is subjected to a tensile stress of 12,000 pounds per square inch parallel to the X axis and a tensile stress of 6,000 pounds per square inch parallel to the Y axis. If E is 30,000,000 pounds per square inch and Poisson's ratio is $\frac{1}{4}$, what is the unit deformation in the direction of each coördinate axis?

	Axis	Unit deformation
Ans.	X	0.00035 elongation.
	Y	0.00010 elongation.
	Z	0.00015 compression.

5. Solve Problem 4 if the unit stress along the Y axis is compression.

Ans.	X	0.00045 elongation.
	Y	0.00030 compression.
	Z	0.00005 compression.

When *biaxial* loads are applied as in Problems 4 and 5 the unit deformation may be greater or less than that due to a single load. In Problem 5 the unit elongation in the direction of the X axis due to the tensile stress of 12,000 pounds per square inch is 0.00040.

* There is no definite agreement as to the symbol for Poisson's ratio. Some writers use $\frac{1}{m}$, others use ν or ρ or μ.

The unit compression along the Y axis due to the load of 6,000 pounds per square inch is 0.00020, which with Poisson's ratio equal to $\frac{1}{4}$ makes an additional unit elongation of 0.00005 along the X axis; so that the total unit elongation along the X axis becomes 0.00045. According to *Saint Venant's* law* the entire unit deformation in any direction due to any combination of stresses should not exceed the unit deformation due to the allowable unit stress. In Problem 5, the unit elongation parallel to the X axis is the same as would be produced by a single unit stress of 13,500 pounds per square inch in the direction of that axis.

If, in Problem 5, the unit compressive stress parallel to the Y axis were also 12,000 pounds per square inch the unit elongation along the X axis would be equivalent to that which would be produced by $12{,}000 + \dfrac{12{,}000}{4}$ or 15,000 pounds per square inch.

The unit stress which is equivalent to that which will produce a deformation equal to the deformation caused by a combination of stresses, may be calculated directly by simply multiplying each unit stress by Poisson's ratio and adding it, with the proper sign, to the other stresses.

Example

A block of metal is subjected to a compressive stress of 8,000 pounds per square inch parallel to the X axis, a tensile stress of 6,000 pounds per square inch along the Y axis, and a compressive stress of 5,000 pounds per square inch along the Z axis. Find the unit stress along each axis which will be equivalent to the stress which gives the deformation which is given by this combination.

Ans. X axis—Equivalent unit stress $= 8{,}000 + 1{,}500 - 1{,}250 = 8{,}250$ lb./in.^{2}

Problems

6. A rod of material, for which Poisson's ratio is $\frac{1}{4}$ and the allowable unit tensile stress is 1,500 pounds per square inch, is subjected to a transverse compression of 2,400 pounds per square inch and a pull in the direction of its length. What is the maximum allowable pull?

Ans. 900 pounds per square inch.

7. The rod of Problem 6 is subjected to a transverse compression of 2,000 pounds per square inch in one direction and a second transverse compression of 1,200 pounds per square inch at right angles to the first. Find the maximum allowable pull in a direction at right angles to the plane of these compressive stresses.

* Saint Venant's law and other theories in regard to the allowable unit stress are discussed further in Chapter XVIII.

14. Volume Change and Modulus of Elasticity.—When a solid is subjected to stress in one direction there is a slight change in volume. If a body of unit dimensions is elongated an amount δ by an external pull, its length becomes $1 + \delta$ and its transverse dimensions become $1 - \sigma\delta$ where σ is Poisson's ratio.

$$\text{Area of cross-section} = (1 - \sigma\delta)^2 = 1 - 2\,\sigma\delta + (\sigma\delta)^2. \tag{1}$$

Since $\sigma\delta$ is small, being never greater than 0.001, its square, which is relatively much smaller, may be neglected, so that, approximately,

$$\text{Area of cross-section} = 1 - 2\,\sigma\delta. \tag{2}$$

Multiplying by the length,

$$\text{Volume} = (1 - 2\,\sigma\delta)(1 + \delta) = 1 + (1 - 2\,\sigma)\delta - 2\sigma\delta^2, \tag{3}$$

of which the last term, $2\,\sigma\delta^2$, may also be neglected, so that

$$\text{Approximate volume} = 1 + (1 - 2\,\sigma)\delta. \tag{4}$$

Subtracting the original volume of one cubic unit,

$$\text{Increment of volume} = (1 - 2\,\sigma)\delta. \tag{5}$$

These formulas apply only to the temporary deformations below the elastic limit. For the permanent change of form which occurs when the elastic limit is exceeded there is practically no change of volume.

Problems

1. A 1-inch cube is stretched 0.0008 inch. If Poisson's ratio is 0.26, how much is the area diminished and how much is the volume increased? Solve without using formulas. *Ans.* 0.000416 square inch decrease.
0.000384 cubic inch increase.

2. A steel bar 2 inches square is subjected to a compression of 60,000 pounds. If E is 30,000,000 pounds per square inch and Poisson's ratio is 0.27, find the decrease in volume per cubic inch and the total decrease in a length of 5 inches.

A more important case is one in which a solid is submerged in a liquid under pressure, which gives it a compressive stress in all directions, and produces a decrease in volume. The quotient obtained by dividing the unit pressure by the relative decrease in volume is called the modulus of volume elasticity. If the pressure of 3,500 pounds per square inch reduces the volume of originally 1 cubic inch to 0.9995 cubic inch, the modulus of volume elasticity is given by

$$E_v = \frac{3{,}500}{0.0005} = 7{,}000{,}000 \text{ pounds per square inch.}$$

Problems

3. A block of steel has its volume changed from 8.440 cubic inches to 8.436 cubic inches by a pressure of 10,000 pounds per square inch. Find the modulus of volume elasticity.

Ans. 21,100,000 pounds per square inch.

4. A copper cylinder for which E_v equals 16,000,000 pounds per square inch is sunk in water to the depth of 24,000 feet. How much is its volume diminished?

The modulus of volume elasticity may be calculated from E and Poisson's ratio. If a cube of unit dimensions be subjected to unit pressure s in the direction of any axis, it is shortened $\frac{s}{E}$ in the direction of the pressure and is elongated $\frac{\sigma s}{E}$ along each of the two axes at right angles to the direction of the pressure. When there is a compressive stress s in every direction, the compression along any axis is made up of the direct compression $\frac{s}{E}$ due to the pressure in that direction and two elongations of $\frac{\sigma s}{E}$ each due to the pressure along the other axes at right angles to the first.

$$\text{Total compression} = \frac{s}{E} - \frac{2\sigma s}{E} = \frac{s}{E}(1 - 2\sigma). \tag{6}$$

The length of each edge of the cube becomes $1 - \frac{s}{E}(1 - 2\sigma)$, and

$$\text{Final volume} = \left\{1 - \frac{s}{E}(1 - 2\sigma)\right\}^3 = 1 - \frac{3s}{E}(1 - 2\sigma) + \frac{3s^2}{E^2}(1 - 2\sigma)^2 +, \text{etc.} \tag{7}$$

Dropping the terms containing the higher powers of $\frac{s}{E}$, and subtracting the original volume of one cubic unit,

$$\text{Decrease in volume} = \frac{3s}{E}(1 - 2\sigma). \tag{8}$$

Dividing the unit pressure s by the unit decrease in volume to get the modulus of volume elasticity,

$$E_v = \frac{s}{\frac{3s}{E}(1 - 2\sigma)} = \frac{E}{3(1 - 2\sigma)}. \tag{9}$$

Table III is taken from experiments by E. H. Amagat (*Annales de Chimie et de Physique*, 1891, page 118).

TABLE III.—LINEAR AND VOLUME ELASTICITY AND POISSON'S RATIO

Material	Modulus of vol. elasticity, E_v, lb./in.²*	Modulus of elasticity, E, lb./in.²		Poisson's ratio, σ	
		1st method	2d method	1st method	2d method
Steel........	21,670,000	29,890,000	30,072,000	0.2694	0.2679
Copper......	17,153,000	17,608,000	18,099,000	0.3288	0.3252
Brass.......	15,424,000	15,700,000	16,202,000	0.3305	0.3236
Lead........	5,324,000	2,390,000	2,195,000	0.4252	0.4313
Flint glass...	6,112,000	9,176,000	....	0.2499	....
Glass.......	6,691,000	10,280,000	....	0.2451	....

Problems

5. Show that if Poisson's ratio is ¼, the modulus of volume elasticity is two-thirds of the modulus of linear elasticity.

6. If E for hard steel is 30,000,000 pounds per square inch, and Poisson's ratio is 0.27, find the modulus of volume elasticity.

7. Taking the values of Poisson's ratio and E from Table III, calculate E_v for copper and lead.

Miscellaneous Problems

1. A stick of Douglas fir tested in tension at the Watertown Arsenal ("Tests of Metals," 1896, page 405) showed an elongation of 0.0427 inch in a gage length of 200 inches when the load per square inch changed from 100 pounds to 500 pounds. Find E.

Ans. 1,874,000 pounds per square inch.

2. A second stick of Douglas fir tested in tension (1896, pages 407–09) showed an elongation of 0.1015 inch in a gage length of 200 inches, and a decrease of width of 0.0020 inch in a width of 12 inches when the load changed from 100 pounds to 1,000 pounds per square inch. Find the modulus of elasticity in tension parallel to the grain and Poisson's ratio.

Ans. Poisson's ratio, 0.33.

3. In a compressive piece cut from the stick of Problem 2, when the compressive stress changed from 100 pounds to 1,000 pounds per square inch, there was a compression of 0.0230 inch in a gage length of 50 inches. Find E_c.

4. A white-oak stick 11.98 inches by 9.95 inches tested in compression (1896, page 425) was shortened 0.0140 inch in a gage length of 50 inches when the load was increased from 11,920 pounds to 71,520 pounds. Find E.

5. A block of the same oak used in Problem 4 was tested in compression across the grain. When the unit stress changed from 20 pounds per square inch to 320 pounds per square inch, the compression in a gage length of 6 inches was 0.0091 inch. Find the modulus of elasticity of oak across the grain.

Ans. 198,000 pounds per square inch.

* In reducing pressure in atmospheres to pounds per square inch, I have used 14.7 pounds per square inch equal 1 atmosphere.

6. Two blocks of Douglas fir were tested in compression across the grain. In the first block the compression was normal to the growth rings, and the compression in a gage length of 6 inches when the unit load changed from 20 pounds to 300 pounds was 0.0081 inch. In the second block the compression was tangent to the growth rings, and the compression in 6 inches with the same change of load was 0.0195 inch. Find E for each case ("Tests of Metals," 1896, pages 396–97).

7. A steel tape 400 feet long hangs vertically downward in a mine shaft. If E is 30,000,000 pounds per square inch, how much is it stretched by its own weight? Solve by means of the average load and check by integration.

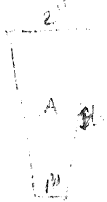
Ans. 0.1088 inch.

8. A round steel rod tapers gradually from 2 inches in diameter to 1 inch in diameter in a length of 20 inches. If E is 30,000,000 pounds per square inch, and if it is assumed that the unit stress in any transverse section is uniform throughout the section, calculate, by means of integral calculus, the elongation of the 20 inches due to a pull of 15,000 pounds. Check the result roughly by comparing with the elongation of uniform rods of 1 inch and 2 inches diameters respectively.

Ans. 0.00637 inch.

9. A plate of uniform thickness t has a breadth b at one end of a given length l and a breadth c at the other end. Find the expression for the elongation of this length l due to a pull P

Ans. $\dfrac{Pl}{Et(c-b)} \log \dfrac{c}{b}$.



CHAPTER II

STRESS BEYOND THE ELASTIC LIMIT

15. Stress-strain Diagrams.—In Chapter I the only unit stresses considered are below the elastic limit. Within that limit unit stress is proportional to unit deformation, and Formula I and the equations of Article 14 hold good. Unit stresses below the elastic limit are the most important from the standpoint of the engineer, for in well-designed structures the unit stress seldom exceeds one-half this limit. It is desirable, however, to know what takes place above the elastic limit and the character of the final failure of the material. To secure this information, tests are made in which a series of loads are applied to a piece of the material in question, and the corresponding deformations are observed with suitable measuring apparatus. Table IV gives a part* of the results of a tension test of a rod of† machine steel. The rod was originally 20 inches long and turned to a diameter of 1.31 inches. About 9 inches of the rod at the middle was turned down further to a diameter of 1.115 inches. A length of 8 inches in this middle portion was taken as the gage length from which to measure elongations. The rod *I* on the right in Fig. 6 (photographed from a rod exactly like the one tested) shows the original form of this test piece. The elongations in this gage length were measured by an extensometer reading to 0.0001 inch (see Johnson's "Materials of Construction," Fig. 271). As there are two micrometers in this extensometer, we are warranted in giving the gage readings to 0.5 of a division. When the load reached 78,000 pounds per square inch, the extensometer was removed and the elongations taken with an ordinary steel scale reading in hundredths of an inch. After fracture the rod was taken from the testing machine, the two

* Readings were taken at 2,000-pound intervals from 56,000 to 76,000 pounds per square inch, and were used in locating the curve of Fig. 6. Readings were also taken at 2,000-pound intervals between 30,000 and 40,000 pounds per square inch, as it was suspected that the yield point might fall between these limits.

† An analysis of this steel, made by Prof. D. J. Demorest, gave: carbon, 0.42 of 1 per cent.; manganese, 0.71 of 1 per cent. The rod was turned from a bar of hot-rolled steel.

Table IV.—Tension Test of Machine Steel

Diameter, 1.115 inches; area of section, 0.976 square inch; gage length, 8 inches

Applied load		Elongation	
Total	Per square inch	In gage length	Per inch length
Pounds	Pounds	Inch	Inch
0	0	0	0
2,926	3,000	0.00085	0.00011
4,880	5,000	0.00145	0.00018
9,760	10,000	0.00260	0.00033
14,640	15,000	0.00410	0.00051
19,520	20,000	0.00535	0.00067
24,400	25,000	0.00665	0.00083
29,280	30,000	0.00795	0.00099
34,160	35,000	0.00920	0.00115
39,040	40,000	0.01075	0.00134
40,992	42,000	0.0114	0.00142
42,944	44,000	0.0144	0.00180
44,896	46,000	0.0356	0.00445
44,000	45,080	0.0734	0.00917
44,500	45,504	0.0965	0.01206
45,000	46,100	0.0973	0.01216
45,872	47,000	0.0981	0.01226
46,848	48,000	0.0991	0.01239
47,824	49,000	0.1013	0.01266
48,800	50,000	0.1163	0.01454
50,752	52,000	0.1273	0.01589
52,704	54,000	0.1381	0.01726
54,656	56,000	0.1552	0.01940
64,416	66,000	0.2601 (1)*	0.03251
74,176	76,000	0.4244 (2)	0.05305
76,128	78,000	0.50 (by	0.0625
78,080	80,000	0.59 scale)	0.0740
79,056	81,000	0.70	0.0875
80,032	82,000	0.76	0.095
81,008	83,000	0.85	0.106
81,984	84,000	0.99	0.124
83,000	85,040	1.24	0.155
83,200	85,240	1.50	0.187
82,000	84,100	1.64 (3)	0.205
80,000	82,000	1.85 (4)	0.231
72,000	73,800 (broke)	1.99 (5)	0.247

* (1) Diameter, 1.097 inches. (2) Diameter, 1.083 inches.
(3) Begins to "neck." (4) Diameter of neck, 0.904 inch.
(5) Elongation measured after fracture. Diameter of neck, 0.821 inch.
Steel, hot-rolled; carbon, 0.42 per cent.

portions placed together as shown in Fig. 6, II, and the final elongation of 1.99 inches measured. Loads were applied and measured by means of a 100,000-pound Olsen testing machine (see Johnson's "Materials of Construction," Fig. 256).

In order to present the results of such a test visually, it is convenient to use the unit stress and the unit elongation as the coördinates in a curve called the *stress-strain diagram,* or simply *stress diagram.*

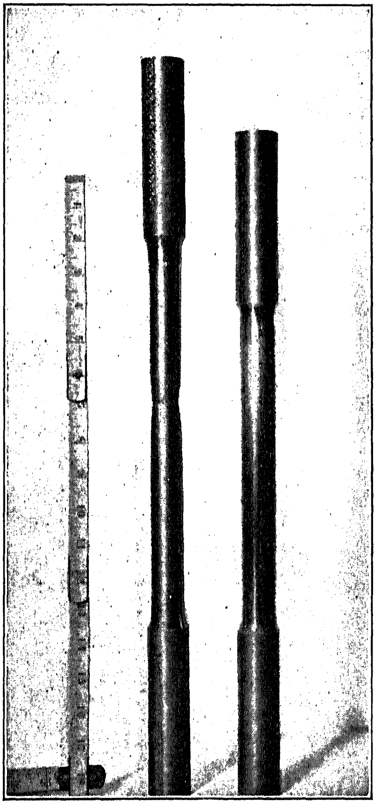

II I

FIG. 6.—Steel rod tested in tension.

In America, the unit stress in pounds per square inch is used as ordinate, and the unit deformation is taken as abscissa. In England, some writers use unit stress as abscissa and unit deformation as ordinate.

Fig. 7 is the stress-strain diagram plotted from Table IV. One division on the horizontal scale represents a unit elongation of 0.01, and one division on the vertical scale represents a unit stress of 5,000 pounds per square inch.

Fig. 8 is a part of the stress-strain diagram from the same table plotted on an enlarged scale; one division on the horizontal represents a unit elongation of 0.0002 inch per inch of length (one-fiftieth as much as in Fig. 7); one division on the vertical represents a unit stress of 2,500 pounds per square inch (one-half as much as in Fig. 7).

16. Elastic Limit and Yield Point.—In Article 8, *the elastic limit* is defined as the maximum unit stress to which a body may be subjected without permanent deformation. *Elastic limit* is also defined as the unit stress at which the stress-strain diagram begins to deviate from a straight line. Defined in this way, it

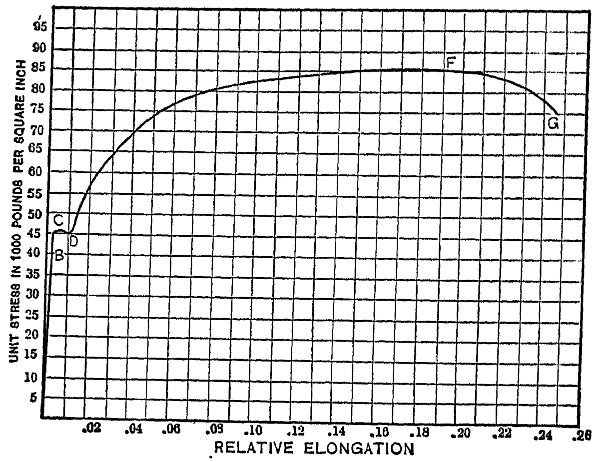


FIG. 7.—Stress-strain diagram of machine steel.

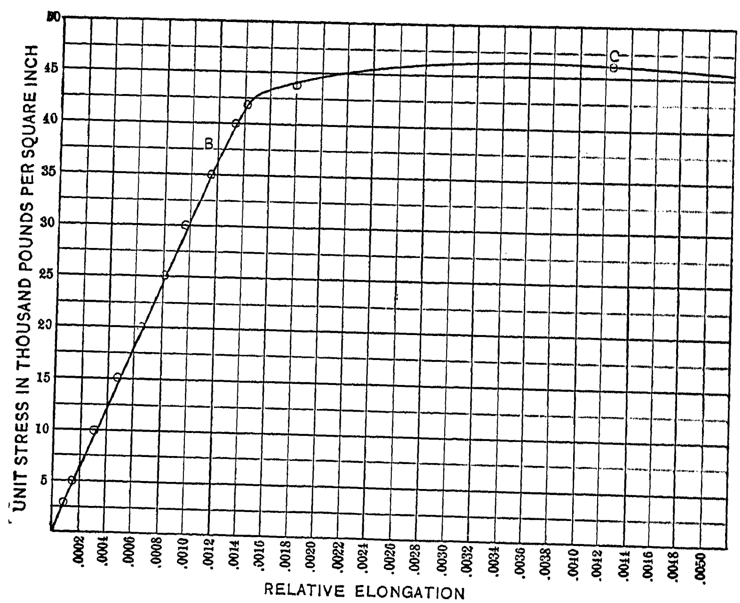


FIG. 8.—Part of diagram for machine steel

is called the *proportional elastic limit.** These two definitions give practically the same unit stress, but it is difficult to determine it accurately from either, owing to the facts that the curve, at first, deviates very slowly from the straight line; the form of the curve depends considerably upon the rate of application of the load; and there are frequently small deformations which vanish slowly after the removal of the external force.

The point B, of Figs. 7 and 8, is the proportional elastic limit. From Table IV, for unit stresses below 35,000 pounds per square inch, the increase in elongation in the gage length is seen to be about 13 divisions for each increment of 5,000 pounds per square inch. Between 35,000 and 40,000 pounds per square inch, the elongation in the gage length is 15.5 divisions. From 40,000 to 42,000 pounds per square inch, the stretch is 6.5 divisions, which is equivalent to 16.2 divisions for a 5,000-pound increase of unit stress. Between 42,000 and 44,000 pounds per square inch, the elongation is 30 divisions, and the rate of increase is more than five times what it is below the proportional elastic limit. From these figures it is evident that the increase in deformation between 35,000 and 40,000 pounds per square inch is not due to an accidental error in measurement, but that there is a definite change in the rate of deformation at some stress slightly above 35,000 pounds per square inch.

At C, at a unit stress of 46,000 pounds per square inch, the curve becomes horizontal. This is the *yield point.* Beyond the yield point the curve drops to a unit stress of about 45,000 pounds per square inch. Not only is there an increase of length with no increase of stress, but there is a considerable elongation with a diminished stress. In changing down to 45,000 pounds and back again to 45,500 pounds, the increase in length is nearly twice as great as the entire elongation up to the yield point, and five times as great as the elongation from zero load to a stress of 42,000 pounds per square inch.

The unit stress at the yield point may easily be determined in rapid commercial tests and without delicate apparatus for measuring the elongations. If we consider Table IV, we find that the total elongation in the gage length of 8 inches is about $\frac{1}{70}$ inch at a unit stress of 44,000 pounds, and rises to more than $\frac{1}{30}$ inch at a unit stress of 46,000 pounds. This increase in length may easily be measured with an ordinary scale, so that

* Frequently called Proportionality Limit.

the yield point may be determined within 1,000 or 2,000 pounds without the use of any extensometer whatever. Again, just beyond the yield point the elongation is increased with a diminished load. This may easily be determined in rapid commercial tests in which the testing machine is kept running continuously. Before reaching the yield point, the poise on the beam of the weighing apparatus must be continually moved out to preserve a balance, showing that the stress is increasing with the elongation. At the yield point the "beam drops" while the elongation increases, and the poise must be moved backward to secure a balance. In iron or steel which has not been turned or polished, and is therefore covered with a coat of oxide, the yield point may be determined by this oxide breaking loose and falling. We sometimes see a portion of a rod reach the yield point before the remainder; the oxide falls from this portion, while the other parts of the bar are unchanged till the stress becomes a little greater. The curve in such a rod will show several steps or bends beyond the first yield point, corresponding to the yield points of the various portions.

Owing to the fact that the yield point may be determined so easily, by methods which were in use before delicate extensometers were available, the term "*elastic limit*" is commonly applied to what is really the *yield point*. When the term "*elastic limit*" is used in specifications, *yield point* is frequently meant. The present tendency is to employ the term *elastic limit* to mean the proportional elastic limit. The late J. B. Johnson suggested the term *true elastic limit*, but this has not come into general use.

17. Johnson's Apparent Elastic Limit.—Since it is somewhat difficult to determine the proportional elastic limit accurately, especially in hard steel, where there is a wide range between this stress and the yield point, and in materials (such as cast iron) which have no yield point, the late Prof. J. B. Johnson proposed another point which he called the "*apparent elastic limit.*"* He defined the apparent elastic limit as "the point on the stress diagram at which the rate of deformation is 50 per cent. greater than at the origin." It is that point on the curve at which the slope of the tangent from the *vertical* is 50 per cent. greater than that of the straight-line part of the curve.

This term has not yet come into general use among engineers.

* See JOHNSON's "Materials of Construction," pages 18–20.

In some investigations of the strength of materials, it has been found useful in comparing the results of different tests.*

18. Calculation of the Modulus of Elasticity.—The stress-strain diagram, when plotted to a sufficiently large scale, enables us to calculate quickly the *average* value of the modulus of elasticity. If the straight line passes through the origin, we merely find the value of the unit stress which corresponds to some convenient unit elongation, such as 0.001 or 0.0005. If the straight line does not pass through the origin, we take the difference of unit stress for some convenient difference of elongation. In either case, to get the modulus, we divide the *difference* in unit stress by the corresponding *difference* in unit elongation.

Problems

1. From the curve of Fig. 8 find the unit stress which corresponds with the unit elongation of 0.0008 and compute E to three significant figures.

2. From Fig. 8 find the unit elongation which accompanies the unit stress of 25,000 pounds per square inch and calculate the modulus of elasticity to three significant figures.

3. From the data of Table IV plot the stress-strain diagram up to the unit stress of 42,000 pounds per square inch to the scales 1 inch equals a unit stress of 5,000 pounds per square inch and a unit elongation of 0.0002 inch per inch of length. Use paper ruled in 0.1-inch units. Draw the curve as a light line and solve Problems 1 and 2.

4. From Table IV calculate E, using intervals of 15,000 pounds per square inch.

The stress-strain diagram gives a convenient means of finding a fair *average* value of the modulus of elasticity from a single calculation. It enables us to judge of the accuracy of the test by observing how closely the points approach the straight line.

If the modulus of elasticity is computed direct from the readings, and considerable accuracy is desired, it is best to use the average of several values taken with equal intervals of unit stress. As the errors in reading the extensometer and setting the scale beam are practically constant, the *relative* errors are inversely proportional to the length of the intervals of stress and deformation;

* See work of H. F. MOORE in *Bulletin* No. 42 and ALBERT J. BECKER in *Bulletin* No. 85 of the University of Illinois Engineering Experiment Station.

consequently fairly large intervals should be used. From Table IV, using an interval of 20,000 pounds per square inch:

Interval	Elongation in gage length in inches	Modulus in pounds per square inch
0–20,000	0.00535	29,900,000
5,000–25,000	0.00520	30,800,000
10,000–30,000	0.00535	29,900,000
15,000–35,000	0.00510	31,400,000
	Average E	30,500,000

It is not necessary to divide out for the unit elongation. The work may be indicated.

$$20{,}000 \div \frac{0.00535}{8} = \frac{20{,}000 \times 8}{0.00535} = 29{,}900{,}000.$$

On the other hand, it is best to divide the total stress by the area (at least for the largest reading used) since it is desirable to know for what range of unit stress a given modulus of elasticity holds.

Problems

5. Using an interval of 20,000 pounds per square inch, what error in E would be produced by an error of one division in the extensometer reading?

19. Ultimate Strength and Breaking Strength.—The point F at the top of the curve of Fig. 7, representing a unit stress of a little more than 85,000 pounds per square inch, gives the *ultimate strength* of the steel under test. The rod at this stress was elongated 1.5 inches in the gage length of 8 inches, and the diameter was practically uniform throughout this length. Beyond this elongation, the rod began to "neck;" its diameter decreasing rapidly at *one section*, while the remainder was not changed. When the load had dropped to about 82,000 pounds per square inch, the minimum diameter at the neck was 0.904 inch, while that of most of the gage length was a little over 1 inch. It finally broke at a total load of 72,000 pounds, which, in terms of the original area, corresponds to a unit stress of 73,800 pounds per square inch. This is the *breaking strength*, the point G of Fig. 7.

Most materials, such as wood, cast iron, concrete, and hard steel, do not neck; the ultimate strength corresponds with the breaking strength.

20. Percentage of Elongation and Reduction of Area.—In ductile materials, such as wrought iron and steel, the percentage of elongation is an important factor. In the tested bar of Table IV

and Fig. 7, the final elongation in 8 inches was 24.7 per cent. The greatest relative elongation is at the neck. To show the variation in elongation, the gage length was subdivided by punch marks into 1-inch spaces. After rupture, the two pieces were placed together as shown in Fig. 6, II, and these spaces were measured with the following results:

Interval	Elongation
0–1	0.17 inch
1–2	0.19 inch
2–3	0.31 inch
3–4	0.54 inch; included neck
4–5	0.25 inch
5–6	0.19 inch
6–7	0.17 inch
7–8	0.17 inch

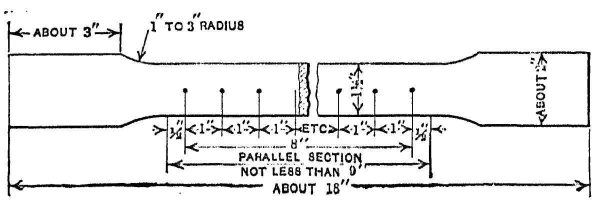


FIG. 9.—Tension test bar—8-inch gage length.

If we use only the interval 3–4, which included the neck, we get an elongation of 54 per cent. If we take the 4-inch interval 0–4, we get 30.2 per cent. In order to make the results of different tests comparable with one another, the Society for Testing

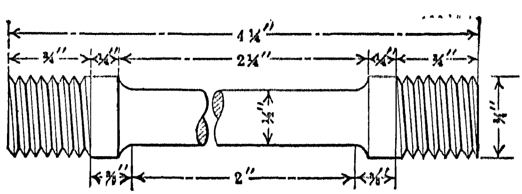


FIG. 10.—Tension test bar—2-inch gage length.

Materials has adopted 8 inches as the standard gage length. Fig. 9 shows the dimensions of the standard test bar of this length, as made from a plate.

From forgings, castings, and other material, from which it is not practicable to take a test bar 18 inches long, pieces are made as shown in Fig. 10, with a 2-inch gage length. In Table V, it will be noted that the percentage of elongation is sometimes given for both gage lengths.

The percentage of reduction of area at the neck is also important. In the rod of Table IV, the original diameter was 1.115 inches and the final diameter at the neck was 0.821 inch. The final area of the neck was 54.2 per cent. of the original area, and the reduction of area was 45.8 per cent.

TABLE V.—SPECIFICATIONS FOR ROLLED STEEL ADOPTED BY AMERICAN SOCIETY FOR TESTING MATERIALS, YEAR BOOK, 1914

Material		Tensile strength, lb. per sq. in.	Yield point, lb. per sq. in.	Elong. in 8 in., per cent.	Elong. in 2 in., per cent.	Reduction of area, per cent.
Steel for bridges.	Structural	55,000 to 65,000	0.5 tens. str.	$\frac{1,500,000}{\text{tens. str.}}$	22	
	Rivet	46,000 to 56,000	0.5 tens. str.	$\frac{1,500,000}{\text{tens. str.}}$		
Structural nickel steel.	Rivet	70,000 to 80,000	45,000	$\frac{1,500,000}{\text{tens. str.}}$	...	40
	Plates, shapes and bars	85,000 to 100,000	50,000	$\frac{1,500,000}{\text{tens. str.}}$	...	25
	Eye-bars and and pins, annealed	90,000 to 105,000	52,000	20	20	35
Billet-steel for concrete reinforcement.	Plain bars, struct. grade	55,000 to 70,000	33,000	$\frac{1,400,000}{\text{tens. str.}}$		
	Plain bars, hard grade	80,000 minimum	50,000	$\frac{1,200,000}{\text{tens. str.}}$		
	Deformed bars, hard grade	80,000 minimum	50,000	$\frac{1,000,000}{\text{tens. str.}}$		

Problems

1. From the above measurements, find the percentage of elongation for the interval 2–5, and also for the interval 1–7.

Ans. 37 per cent., 27.5 per cent.

2. A rod of soft steel, originally 0.874 inch in diameter, was tested in tension. After a fracture under a final load of 23,400 pounds, the gage length of 8 inches was found to be 10.98 inches, and the diameter at the neck was 0.499 inch. Find the percentage of elongation and the reduction of area.

Ans. 37.2 per cent. elongation.
67.4 per cent. reduction of area.

3. The maximum load in Problem 2 was 32,850 pounds. Find the tensile strength. *Ans.* 54,750 pounds per square inch.

4. From Table V, what should be the relative elongation in structural steel for bridges having a tensile strength of 60,000 pounds per square inch? *Ans.* 25 per cent.

5. What should be the relative elongation of hard grade deformed bars for reinforcement for concrete, having an ultimate strength of 88,000 pounds per square inch, in order to satisfy the minimum requirements of the specifications of the Society for Testing Materials? *Ans.* 11.4 per cent.

21. Apparent and Actual Unit Stress.—The unit stresses of Table IV were calculated by dividing the total load by the area of the cross-section at the beginning of the test. This is the usual custom and stresses are always so understood unless otherwise designated. Owing to the permanent reduction of area in a ductile material after passing the yield point, the *actual unit stress*, which is calculated by dividing the total load by the actual area of cross-section when loaded, may be much larger. In the bar of Table IV, the actual diameter, when the load was 74,176 pounds, was 1.083 inches, and the actual area of cross-section was 0.921 square inches. The actual unit stress was the quotient of 74,176 divided by 0.921 which is 80,540 pounds per square inch, while the apparent unit stress was only 76,000 pounds per square inch.

Problems

1. From Table IV calculate the actual unit stress at the neck for the last two loads.

2. From Table IV calculate the actual unit stress when the apparent unit stress was 66,000 pounds per square inch.

Before necking begins the actual unit stress may be calculated from the apparent unit stress and the relative elongation. The volume of the gage length remains nearly constant. If A represents the original area of cross-section, the volume of a portion 1 inch in length is equal to A cubic inches. If A' is the area of cross-section when the original inch length is stretched to a length $1 + \delta$, the volume is $A'(1 + \delta)$.

$$A = A'(1 + \delta), \quad A' = \frac{A}{1 + \delta}.$$

$$\text{Actual unit stress} = \frac{P}{A'} = \frac{P}{A}(1 + \delta).$$

Actual unit stress = apparent unit stress multiplied by $(1 + \delta)$.

Problems

3. From Table IV calculate the actual unit stress when the apparent unit stress is 82,000 pounds per square inch.

4. From Table IV calculate the actual unit stress when the apparent unit stress is 40,000 pounds per square inch.

Fig. 11 shows the actual and apparent unit stress diagrams for a rod of soft steel. While the curve of apparent unit stress

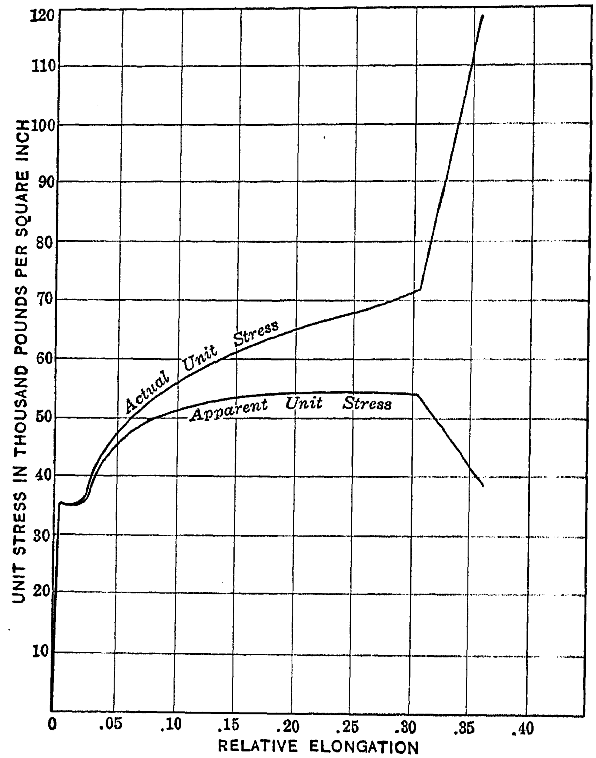


FIG. 11.—Actual and apparent unit stress.

drops when the rod begins to neck, the curve of actual unit stress rises at all points except at the yield point.

An apparent discrepancy may be noticed between the statements of this article and those of Article 14. Poisson's ratio and the theory of Article 14 apply only to the *temporary deformations* inside the elastic limit, while the statement that the volume remains constant applies to the *permanent deformations* beyond the yield point. While the bar is under load, there is also the elastic deformation superimposed on the permanent deforma-

tion, but the permanent deformation is the principal factor. In so far as the permanent deformation is concerned, the material beyond the yield point behaves as if Poisson's ratio were ½.

22. Curves of Various Structural Materials.—The curves of Figs. 7 and 8 give a fair average idea of the behavior of machine steel in tension. The apparent stress curve of Fig. 11 shows the

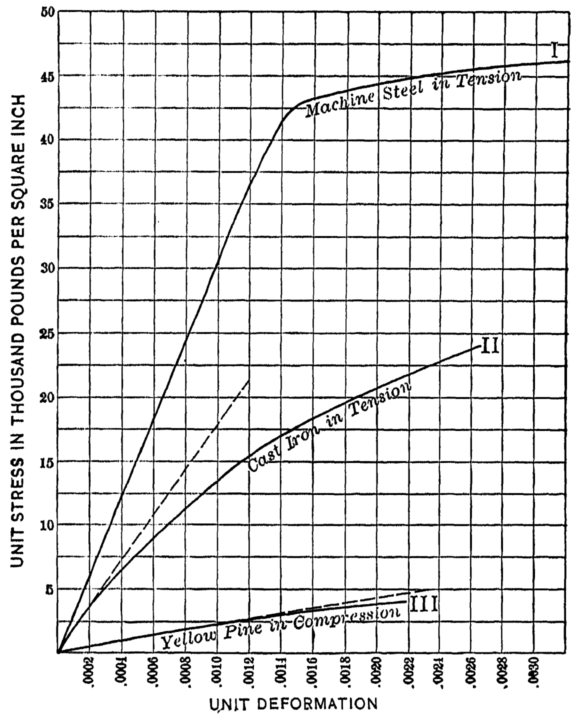


FIG. 12.—Stress-strain diagrams of steel, timber, and cast iron.

same for a sample of rather soft steel. This sample was not analyzed but it probably contained less than 0.15 per cent. carbon. Its tensile strength was a little below the minimum requirement of structural steel for bridges.

The modulus of elasticity of structural steel is about 29,000,000 pounds per square inch and the yield point a little over one-half the tensile strength.

Tool steel, with 1 per cent. or more of carbon, has a tensile strength of over 100,000 pounds per square inch and a modulus of elasticity of 30,000,000 pounds per square inch. The ultimate

strength and yield point of all steel is changed by heat treatment and mechanical treatment, but these have little effect upon the modulus of elasticity.

Table VI and curve II of Fig. 12 represent the behavior of cast iron in tension. The table is the mean of the tests of six bars from the same heat. The figures represent what may be expected in good cast iron.

Table VI is from the average of six tests, specimens 8,014, 8,041, 8,050, 8,051, 8,053, and 8,063, at the Watertown Arsenal ("Tests of Metals," 1905).

TABLE VI.—TENSION TEST OF CAST IRON

Diameter, 1.129 inches; area, 1 square inch; gage length, 10 inches.

Load per square inch	Elongation	
	In gage length	Per inch length
Pounds	Inch	Inch
1,000	0.00056	0.000056
2,000	0.00112	0.000112
3,000	0.00171	0.000171
4,000	0.00236	0.000236
5,000	0.00303	0.000303
6,000	0.00374	0.000374
7,000	0.00446	0.000446
8,000	0.00526	0.000526
9,000	0.00606	0.000606
10,000	0.00691	0.000691
11,000	0.00779	0.000779
12,000	0.00871	0.000871
13,000	0.00968	0.000968
14,000	0.01061	0.001061
15,000	0.01174	0.001174
16,000	0.01283	0.001283
17,000	0.01404	0.001404
18,000	0.01544	0.001544
19,000	0.01689	0.001689
20,000	0.01851	0.001851
21,000	0.02003	0.002003
22,000	0.02182	0.002182
23,000	0.02420	0.002420
24,000	0.02626	0.002626

The average ultimate load was 26,450 pounds per square inch. The actual initial load was 1,000 pounds. The table is calculated on the assumption that the elongation from 0 to 1,000 is the same as from 1,000 to 2,000.

The curve for this cast iron is plotted to the same scale as Fig. 8, and a part of the curve of steel from Fig. 8 is drawn for comparison. The dotted line shows approximately the initial slope of the cast-iron stress diagram. The curve begins to bend almost at the start, and it is difficult to locate the elastic limit. There is no yield point, and the material breaks without necking.

Problems

1. From the dotted line of curve II, Fig. 12, calculate the modulus of elasticity of cast iron. Check results by means of the readings of Table VI.

2. From Table VI find Johnson's apparent elastic limit for cast iron. Find the difference in elongation for each successive 1,000 pounds, and locate the unit stress at which this difference is one-half greater than at the beginning.

Table VII, and curve III of Fig. 12, represent the behavior of long-leaf yellow pine in compression. Like steel, the curve for timber is a straight line for a considerable portion of its length. In other respects it resembles the curve for cast iron. The post represented by Table VII failed outside of the gaged portion; the ultimate deformation is, therefore, less than it would be if the failure had occurred inside of this length.

Problems

3. Plot Table VII to the scale 1 inch equals 1,000 pounds per square inch, and 1 inch equals 0.0005 relative deformation. Find E and the proportional elastic limit.

4. Find E and the elastic limit of yellow pine from Table VII.

Table VII.—Compression Test of Long-leaf Yellow Pine

From Watertown Arsenal Report, 1897, page 420.
Length of post, 10 feet. Dimensions, 9.75 inches by 9.77 inches.
Area, 95.26 square inches. Gage length, 50 inches.

Applied load		Deformation	
Total	Unit stress per square inch	In gage length	Unit per inch length
Pounds	Pounds	Inch	Inch
9,526	100	0.0021	0.000042
19,052	200	0.0044	0.000088
28,578	300	0.0067	0.000134
38,104	400	0.0091	0.000182
47,630	500	0.0116	0.000232
57,156	600	0.0141	0.000282
66,682	700	0.0165	0.000330
76,208	800	0.0191	0.000382
85,734	900	0.0215	0.000430
95,260	1,000	0.0240	0.000480
114,312	1,200	0.0290	0.000580
133,364	1,400	0.0340	0.000680
152,416	1,600	0.0389	0.000778
171,468	1,800	0.0443	0.000886
190,520	2,000	0.0495	0.000990
209,572	2,200	0.0546	0.001092
228,624	2,400	0.0601	0.001202
247,676	2,600	0.0652	0.001304
266,728	2,800	0.0705	0.001410
285,780	3,000	0.0758	0.001516
304,832	3,200	0.0811	0.001622
323,884	3,400	0.0869	0.001738
342,936	3,600	0.0932	0.001864
361,988	3,800	0.1005	0.002010
381,040	4,000	0.1077	0.002154
400,092	4,200	0.1084	0.002168
416,000	4,367	*Ultimate strength*	

Failed by crushing at end.

TABLE VIII.—COMPRESSION TEST OF 1:2½:6 CONCRETE; AGE, 90 DAYS

Diameter of test cylinder, 8 inches; area, 50 square inches. Total length, 16 inches; gage length, 10 inches.

Applied load		Deformation	
Total	Per square inch	In gage length	Per inch length
Pounds	Pounds	Inch	Inch
2,000	40	0.00013	0.000013
4,000	80	0.00026	0.000026
6,000	120	0.00038	0.000038
8,000	160	0.00052	0.000052
10,000	200	0.00068	0.000068
12,000	240	0.00081	0.000081
14,000	280	0.00099	0.000099
16,000	320	0.00113	0.000113
18,000	360	0.00136	0.000136
20,000	400	0.00158	0.000158
22,000	440	0.00180	0.000180
24,000	480	0.00206	0.000206
26,000	520	0.00232	0.000232
28,000	560	0.00260	0.000260
30,000	600	0.00295	0.000295
32,000	640	0.00327	0.000327
34,000	680	0.00377	0.000377
36,000	720	0.00421	0.000421
38,000	760	0.00473	0.000473
40,000	800	0.00535	0.000535
42,000	840	0.00609	0.000609
44,000	880	0.00692	0.000692
46,000	920	0.00796	0.000796
48,000	960	0.00922	0.000922
50,000	1,000	0.01058	0.001058
52,000	1,040	0.01177	0.001177
54,000	1,080	0.01323	0.001323
56,000	1,120	0.01575	0.001575
58,000	1,160	0.01847	0.001847
60,000	1,200	*Failed*	

Fig. 13 gives some comparative curves for timber and concrete. Curve I is the long-leaf yellow pine of Table VII. The unit deformations are represented on a scale twice as great as in Fig. 12, and the unit stresses, by a scale ten times as great. Curve II of Fig. 13, is the stress diagram for a sample of 1 : 2.5 : 6 concrete in compression, the readings for which are given in Table VIII.

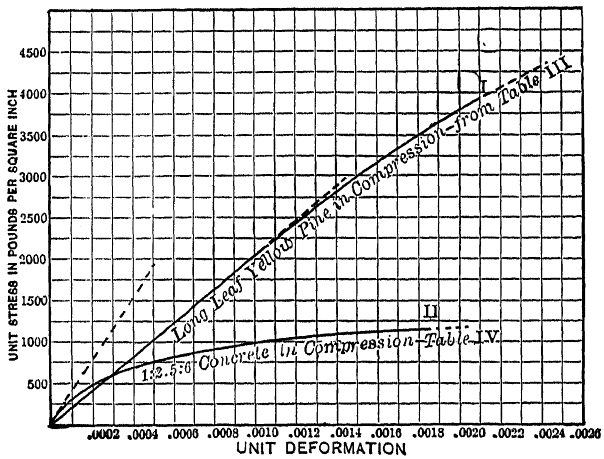


FIG. 13.—Stress-strain diagrams of concrete and yellow pine.

Problems

5. From Table VIII find E for this concrete.

6. Find the difference in elongation for each successive 40 pound increment of unit stress in Table VIII and determine Johnson's apparent elastic limit.

23. Factor of Safety.—In Article 6, the allowable unit stress was defined as depending upon the judgment of some authority. These judgments are based on tests of materials such as those of Tables IV, VI, VII and VIII.

Working stresses should never exceed the elastic limit. They are generally based on the ultimate strength of the material. The ratio of the ultimate strength of a given material to the actual ~~allowable~~ working stress is called the *factor* of *safety.*

Problems

1. If the steel of Table IV is used with a factor of safety of 5, what is the allowable unit stress?

2. A concrete pier 16 inches square carries a load of 75,000 pounds. Using Table VIII, find the factor of safety. *Ans.* 4.

3. Structural steel is used with the allowable unit stress of Table 1. The steel meets the minimum requirement of Table V for steel for bridges. What is the factor of safety based on the tensile strength? What is it based on the yield point?

4. Yellow pine is used with the unit stress of Table I. If Table VII gives the average compressive strength of yellow pine, what is the factor of safety?

The value of the factor of safety which should be used depends upon a great number of conditions. Some of these are:

Repeated stresses slightly beyond the elastic limit will finally cause failure, so that a body subjected to varying load should have its allowable stresses well below this limit. The greater the variation of stress, the smaller should be the allowable unit stress.

The factor of safety must be large enough to allow for any deterioration of the material from any cause during the time which it is to be used. This includes the decay of timber, the rusting of metal, the effect of frost and electrolysis.

In deciding what factor of safety to use, the uniformity of the material must be taken into account. Structural steel which has an ultimate strength of 60,000 pounds per square inch on an average will seldom vary 5,000 pounds on either side of this figure; while the variation of timber sufficiently good to pass a reasonable inspection may be 50 per cent. of the average ultimate strength. An engineer, in designing a concrete structure which he knows will be built under competent supervision, will use much higher unit stresses than he will risk where such inspection is wanting.

The factor of safety must also depend upon the amount of injury which would occur if the material failed. We would use a plank in a scaffold 3 feet high with a much lower factor of safety than we would consider if failure meant a fall of 100 feet.

The factor of safety must allow some margin for unexpected loads. Cases have occurred where a wagon bridge has failed when used as a grandstand to watch a boat race or fireworks. That part of the factor of safety which makes allowance for lack of ordinary judgment in persons using the machine or structure is called the "fool factor."

24. Effect of Form on the Ultimate Strength.—We have assumed in our discussions that the stress across any section is uniform. This is true in a rod of uniform section at some dis-

tance from the surface of application of the load, provided that the line of resultant force coincides with the axis of the rod.

Test bars are made of uniform section throughout, or of uniform section for some distance beyond the extremities of the gage length (see Figs. 6, 9 and 10). Fig. 14 represents one end of such a bar. The stress which may be uniformly distributed across a section at A is unequally distributed at sections B and C, and becomes uniform and parallel to the axis at D. If the gage length began at C at the beginning of the parallel portion, the measured elongation would be too high, owing to the fact that the stress is greater than the average near the surface. This

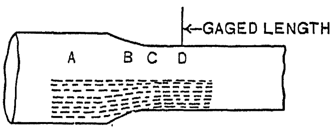


FIG. 14.—Stress distribution in test bar.

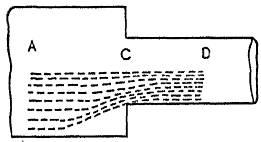


FIG. 15.—Abrupt change of section.

effect would be increased if the change in section were abrupt as in Fig. 15.

The ultimate strength of a rod at such a change of section depends upon its ductility. If rods as in Fig. 15 are made of cast iron or other *nonductile* material, they will fail at *section* C owing to the concentration of stress near the surface. The more abrupt the change the greater the concentration and the easier the failure. If the rod is of *ductile material*, such as structural steel, the strength at C *will be increased* by the material of the larger section to the left. A ductile substance necks before it fails. The material of the larger section tends to prevent necking in the smaller sections at a considerable distance to the right of C.

A rod of ductile material with a short reduced area, such as I and II, Fig. 16, will show a considerably higher ultimate strength than a rod in which the minimum section is longer, as in III, Fig. 16.

It is not necessary to make test bars of the form shown in Figs. 9 and 10; any bar of uniform section will do, and many tests are made of such bars as they come from the rolls. There is this advantage in the standard form shown in Figs. 9 and 10—that it will fail inside the gage length on account of the resistance to neck-

ing for some distance from the larger section. A bar of uniform section may fail outside of the gage length.

It is hardly necessary to state that all changes in section should be gradual. The standard form of bar, as adopted by the Society for Testing Materials (Figs. 9 and 10) changes from large to small section on the arc of a circle tangent to the surface of the smaller section. It is easier to make a *taper* from one size to the other, and the results are practically as good.

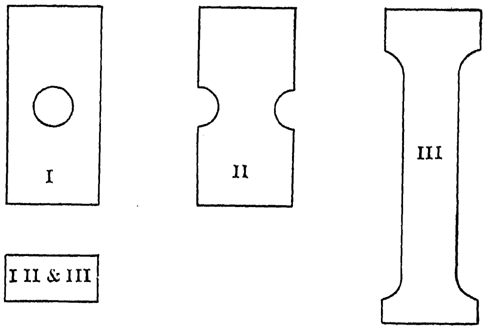


Fig. 16.—Reduced sections.

25. Effect of Stresses beyond the Yield Point.—In materials which are not ductile, any stress beyond the elastic limit produces a permanent injury. In ductile materials, especially soft iron and steel, this is not the case. If a rod of steel or iron, originally hot-rolled, is stressed beyond the yield point, the result is a raising of the yield point. Suppose a rod of soft steel having a yield point of 35,000 pounds per square inch is carried up to 50,000 pounds per square inch, producing a considerable permanent set. Let this rod be again loaded and it will be found to have a yield point of about 50,000 pounds per square inch, the exact value depending somewhat upon the speed at which the two tests are made.

When a high elastic limit and yield point are desired, soft steel is subjected to cold rolling. Fig. 17 shows the effect of cold rolling. The middle rod is a piece of $\frac{7}{8}$-inch cold-rolled shafting. The left one is an exactly similar rod after testing in tension. Its ultimate strength was over 86,000 pounds per square inch, its yield point was about 80,000 pounds per square inch, and its elongation about 10 per cent. On the right is a third rod, originally like the others, which was annealed by heating to redness and

slowly cooling to destroy the effect of the previous cold rolling. When tested in tension, its ultimate strength was found to be 60,000 pounds per square inch, its yield point was 40,000 pounds per square inch, and its elongation 22 per cent. It will be seen from these tests that cold rolling raises the yield point to nearly the ultimate strength and that it increases the ultimate strength a considerable amount.

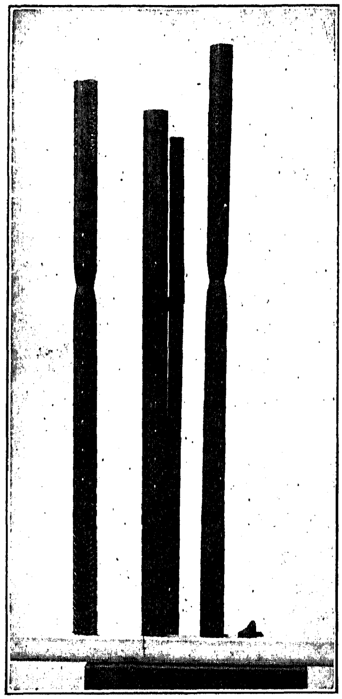


FIG. 17.—Soft steel in tension; left, cold-rolled; right, annealed.

The fact that soft steel may be stressed beyond the yield point without injury, and with no change except a slight reduction of section and elevation of the yield point, is of great advantage in its use in structures. In a heavy structure made of many parts, there is always some adjustment when the loads are first applied. This may cause an overstraining of some parts. If these parts are made of soft steel, they can yield slightly, permitting other members to take part of the excess load.

CHAPTER III

SHEAR

26. Shear and Shearing Stress.—We have learned that when a body is subjected to a pair of forces in the *same line, tensile stress* is produced, if the forces are directed away from each other, and *compressive* stress, if they are directed toward each other.

If the forces are in *parallel* lines or planes, *shearing* and bending stresses are produced in the portion of the body between the planes of the forces. In Fig. 18, the block A is securely held by the body B and a horizontal force P is applied by a second body C. This force P is parallel to the upper surface of B. The body B exerts a horizontal force on the block which is equal and opposite to the force in C. If we consider that portion of the block A between the plane of the upper surface of B and the plane EFG of the lower surface of C, we find that it is subjected to a pair of equal and opposite forces. The material of this portion of the block is subjected to *shearing* and *bending* stresses. The shearing stresses depend upon the magnitude of the forces and the area of the section of A. The bending stresses depend upon these and also upon the distance of the forces apart. If the body C is brought very close to B, so that the distance between the two forces P and P' becomes negligible, the unit bending stress becomes small, while the unit shearing stress is unchanged. The average unit shearing stress is calculated by dividing the force P by the area of the cross-section EFG or the area of any section parallel to it. We notice that in tension or compression we divide the total force by the area of the cross-section *perpendicular* to its direction to get the unit stress; while in shear we divide the total force by the area of the cross-section *parallel* to the forces.

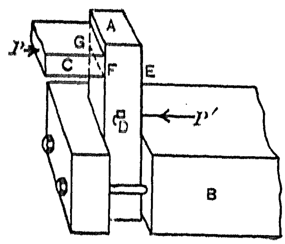


Fig. 18.—Shear and bending.

In this, as in all other cases, the line P in the drawing represents the resultant of a set of forces distributed over an area. The resultant P' must fall some distance below the upper surface of

B, and the resultant P must be above the lower surface at C. It is, therefore, not convenient by this method to get shearing stress entirely free from bending or compressive stress. We will find later that the distribution of shearing stress, when combined with bending, is not uniform over the section; but for the present we shall take no account of this variation, and shall calculate the average unit shearing stress by dividing force by area.

Table IX.—Allowable Unit Shearing Stress

(To be memorized)

Material	Pounds per square inch
Steel shop rivets for bridges	10,000
Steel web plates for bridges	9,000
Steel bolts for bridges	7,000
Steel rivets in boilers	8,800
Yellow pine parallel to grain	125
White oak parallel to grain	200

Problems

1. A 1-inch round rod projects horizontally from a vertical wall. A ring hung on it supports a load of 6,000 pounds. Find the average unit shearing stress in the rod.

Ans. 7,639 pounds per square inch.

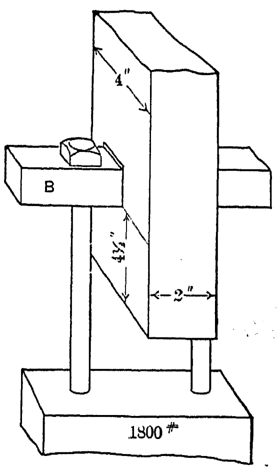


Fig. 19.—Shear in timber

2. A bar 2 inches wide and ¾ inch thick rests on two supports and carries a load of 450 pounds midway between them. Find the average unit shearing stress.

Ans. 150 pounds per square inch.

3. A 4-inch by 4-inch white-oak block has a transverse notch cut in one side, the edge of the notch being 8 inches from the end of the block. A pull of 4,000 pounds is applied by means of a second block set in the notch. Find the unit shearing stress.

Ans. 125 pounds per square inch.

4. In Problem 3, if the grain is parallel to the direction of the pull, is the construction safe? What would be the maximum allowable load? What would it be if the block were yellow pine?

5. A 2-inch by 4-inch long-leaf yellow pine block, hung vertical and supported at the upper end, has a hole 1 inch square perpendicular to the 4-inch faces. The lower edge of this hole is 4½ inches from the lower end

of the block. If the load of 1,800 pounds is hung on a square rod passing through this hole, what is the mean unit shearing stress? What would be the maximum allowable load? (Fig. 19.)

Ans. Maximum allowable load 2,250 pounds.

6. The head of a 1-inch bolt is ¾ inch thick. Find the unit shearing stress tending to strip the head from the bolt when subjected to a pull of 12,000 pounds. *Ans.* 5,093 pounds per square inch.

7. In Problem 6, what is the unit tensile stress in the weakest part of the bolt, if the pull is applied by means of a nut? (See handbook for dimensions.)

27. Shearing Deformation.—Consider a portion D of block A of Fig. 18. The portion extends through the block with its long dimension perpendicular to the plane which contains the resultant P and P''. It is represented on an enlarged scale by the rectangle $HIJK$, Fig. 20. When the shearing forces are

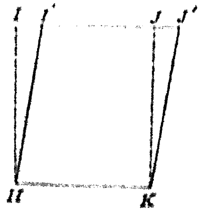


FIG. 20.—Shearing deformations.

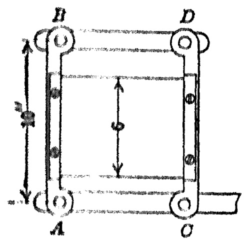


FIG. 21.—Device for illustrating shear.

applied as shown in Fig. 18, it is distorted to the form $HI'J'K$. If we regard HK as fixed, the total displacement of any point in the upper line is equal to II' or JJ'. The unit shearing deformation, which we will represent by δ_s, is the ratio of this horizontal displacement II' to the vertical distance HI. In the case of linear deformation, the unit deformation is obtained by dividing the total deformation by a length in the *same direction as the deformation;* in shearing deformation, the displacement is divided by a distance at *right angles* to the *displacement*. The unit displacement is the tangent of the angle IHI' or JKJ'. The effect of the shearing forces is to lengthen the diagonal HJ, and shorten the diagonal IK.

Problems

1. Two equal bars, AB and CD, Fig. 21, are hinged to a second pair of equal bars, AC and BD, to form a parallelogram. A sheet of rubber, 6 inches wide, has one edge securely clamped to AB and the other edge to CD. The length of AB, center to center of hinges, is 10 inches. What is

the unit shearing displacement when B is displaced 0.2 inch to the right of the vertical? *Ans.* Unit shear, $\delta_s = 0.02$.

2. A hollow circular shaft, 5 inches in diameter, is subjected to a twisting moment, and it is found that two sections, 10 feet apart, suffer a relative displacement of 2 degrees. What is the total shearing displacement of the fibers? What is the unit displacement?

Ans. Total displacement, 0.0873 inch.
Unit displacement, 0.0007275.

28. Modulus of Elasticity in Shear.—The modulus of elasticity in shear is obtained by dividing the unit shearing stress by the unit shearing deformation, just as the modulus of elasticity in tension or compression is computed by dividing the unit tensile or compressive stress by the corresponding unit deformation.

$$E_s = \frac{s_s}{\delta_s}.$$

This modulus of elasticity is frequently called the modulus of rigidity. Forces applied as in Fig. 18 do not give pure shearing stress. It is only in the case of torsion, as in Problem 2 of Article 27, that we get pure shear.

Problems

1. In Problem 2, of Article 27, if E_s equals 12,000,000 pounds per square inch, what is the unit shearing stress? *Ans.* 8,730 pounds per square inch.

2. What is the maximum allowable unit shearing deformation if the modulus of rigidity is 11,000,000 pounds per square inch and the maximum allowable unit shearing stress is 9,000 pounds per square inch?

Ans. 0.000818.

3. A 4-inch shaft 10 feet long is subjected to a twisting moment. One end is fixed. How much may the other end move if the unit stress does not exceed 8,000 pounds per square inch in the outer fibers and E_s = 11,000,000 per square inch? What will be the angle of twist in radians, and what will it be in degrees?

Miscellaneous Problems

1. What is the force required to punch a $\frac{3}{4}$-inch round hole in a $\frac{3}{8}$-inch steel plate, if the ultimate unit shearing strength of the plate is 38,000 pounds per square inch? *Ans.* 33,576 pounds.

2. In Problem 1, what is the unit compressive stress in the punch?

Ans. 76,000 pounds per square inch.

3. If s_c is the ultimate compressive strength of the punch, and s_s the ultimate shearing strength of the plate, and if t is the thickness of the plate, and d is the diameter of the punch, show that the minimum diameter is given by the equation

$$d = \frac{4\,s_s t}{s_c}.$$

4. Apply the result of Problem 3 to find the smallest hole which can be punched in a ¾-inch plate if the ultimate shearing strength of the plate is 40,000 pounds per square inch and ultimate compressive strength of the punch is 160,000 pounds per square inch.

5. In Fig. 22, A and B are short compression members or struts of yellow pine, joined together at the top by a bolt or pin and held from spreading at the bottom by being set into the notches in the bottom chord C. If the load P is 6,000 pounds, what is the unit compressive stress in A and B? What is the maximum unit tensile stress in C? What must be the length of the section d to avoid shearing, if C is made of oak?

Ans. Length of d, 6.5 inches.

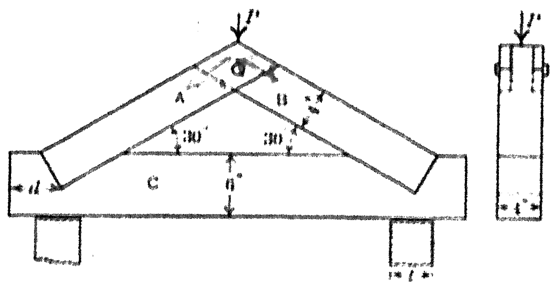


FIG. 22.—Stresses in a truss.

6. In Problem 5, what must be the minimum value of t if the support is yellow pine with the grain horizontal? What would it be if the support is yellow pine with the grain vertical?

29. Shear Caused by Compression or Tension.—Fig. 23 represents a block subjected to a compressive force P in the direction of its length and an equal reaction at the bottom. Imagine the block cut by a plane normal to its length and glued together again. If we consider the portion of the block above the section $BCDE$ as a free body and resolve vertically, we have the force P acting downward equal to the upward reaction of the glued

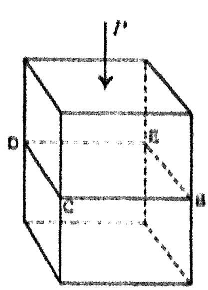


FIG. 23.—Section normal to force.

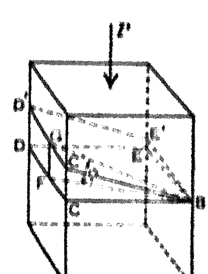


FIG. 24.—Section inclined to force.

surface. (Neglect the weight of the portion above $BCDE$.) If A is the area of the glued surface, the unit compressive stress is given by

$$s_c = \frac{P}{A}.$$

If we resolve horizontally, that is, parallel to any line in $BCDE$, all the components of the external force are zero and the unit shearing stress is zero. If the body was actually made of two portions, the upper portion would not slide on the lower portion, no matter how smooth the surfaces of contact might be.

Now consider a similar body, Fig. 24, cut by a plane $BC'D'E'$ which makes an angle ϕ with the normal plane. Taking the portion above the plane as a free body, as before, we will resolve the external force P perpendicular and parallel to the plane. The total perpendicular component is $P \cos \phi$, and the unit compressive stress is this component divided by the area of the section. If A is the area of the normal section $BCDE$, the area of the inclined section is $A \sec \phi$.

$$\text{Unit compressive stress} = \frac{P \cos \phi}{A \sec \phi} = \frac{P}{A} \cos^2 \phi = \frac{P}{2A}(1 + \cos 2\phi). \quad (1)$$

Resolving parallel to the line BG, which makes the maximum angle with the normal plane, the component of P is equal to $P \sin \phi$. The unit shearing stress is this component divided by the area of the inclined section.

$$s_s = \frac{P \sin \phi}{A \sec \phi} = \frac{P}{A} \sin \phi \cos \phi = \frac{P}{2A} \sin 2\phi. \quad (2)$$

The same relations hold for tension and compression.

Problems

1. Show from equations (1) and (2) that shearing stress is zero and compressive stress a maximum when ϕ is zero.

2. A 2-inch by 4-inch block is subjected to a load of 6,000 pounds in the direction of its length. Find the unit compressive stress and the unit shearing stress with respect to a plane which makes an angle of 25 degrees with the normal section.

Ans. $\begin{cases} s_c, \text{ 616 pounds per square inch.} \\ s_s, \text{ 287 pounds per square inch.} \end{cases}$

3. Solve Problem 2, if the plane makes an angle of 65 degrees with the normal section.

4. A 2-inch by 2-inch white oak post has the grain at an angle of 12 degrees with the direction of its length. What is the maximum allowable load? (See Table IX.) What would be the maximum allowable load if the grain makes an angle of 8 degrees with the direction of its length?

Ans. 3,934 pounds; 4,000 pounds.

5. Prove that the unit shearing stress produced by a single tensile or compressive load is a maximum at 45 degrees with the direction of the load, and that the maximum unit shearing stress is one-half of the tensile or compressive stress which produces it.

30. Shearing Forces in Pairs.—If a body is subjected to pure shearing stress (with no tension or compression except that due to shear), there must be two sets of shearing forces to produce equilibrium and the unit shearing stress must be the same in both. Fig. 25 represents a rectangular block AB with two other

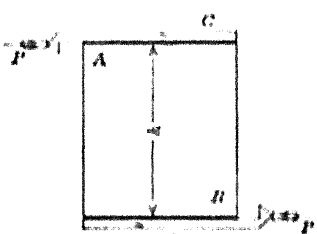


FIG. 25.—Pair of shearing forces.

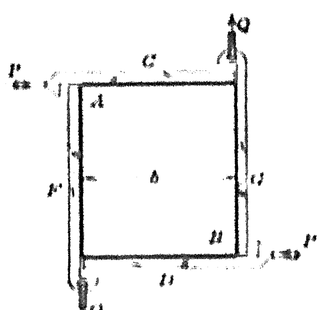


FIG. 26.—Two pairs of shearing forces.

blocks C and D glued to the top and bottom, respectively. There is a horizontal force P, toward the right, acting on the block C and an equal and opposite force acting on the block D. These two forces form a couple tending to rotate the system in a clockwise direction. To produce equilibrium, a block F is glued to the left vertical face of AB (Fig. 26) and a block G is glued to the right vertical face. A downward force Q is applied to F and an equal upward force is applied to G. The breadth of AB is b and its height is h. Equilibrium will occur when the moments of the two couples are equal, that is, when

$$Ph = Qb. \tag{1}$$

The force is transmitted from C and D to AB as a horizontal shear in the glue. Shearing stress is represented by an arrow with a single barb. The arrow in C with barb upward represents the shearing stress from C to AB. If we wished to represent

the opposite shearing stress from AB to C we would place the arrow in AB, pointing toward the left, and with the barb down.

If l is the length of the block AB perpendicular to the plane of the paper, the top and bottom surfaces each have an area bl, and

$$P = sbl, \tag{2}$$

where s is the unit horizontal shearing stress.

The area of each vertical face perpendicular to the plane of the paper is hl and

$$Q = s'hl,$$

where s' is the unit vertical shearing stress.

Since

$$Ph = Qb,$$
$$sblh = s'hlb, \tag{4}$$
$$s = s'. \qquad \text{Formula III.}$$

Formula III applies to any portion of block AB cut out by horizontal and vertical planes perpendicular to the plane of the paper. Fig. 27 represents one such block.

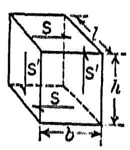


FIG. 27. Equilibrium in shear.

But a block may be held in equilibrium by shearing forces which are not at right angles to each other. Fig. 28 represents one end of a block of width b and vertical *height* h, with one set of forces horizontal, and the other set at an angle ϕ with the horizontal. The slant height of the inclined faces is h cosec ϕ, and the area of each is hl cosec ϕ. The perpendicular distance between them is $b \sin \phi$ so that

$$\text{Moment} = s'hl \operatorname{cosec} \phi \; b \sin \phi = s'hlb. \tag{5}$$

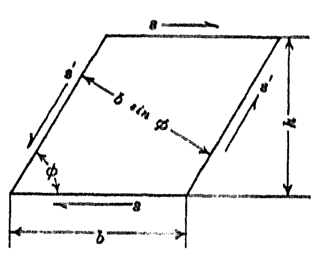


FIG. 28.—Oblique shear.

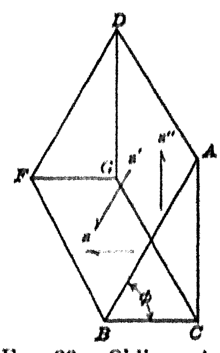


FIG. 29.—Oblique shear.

The moment of the horizontal forces is the same as in the preceding case,

$$sblh = s'hlb,$$
$$s = s'. \qquad \text{Formula III.}$$

Fig. 29 represents a triangular prism cut from Fig. 28 by a vertical plane. We will resolve vertically to find the unit shearing stress, s'', in the vertical face $ACGD$. The total shearing stress in $ABFD$ is $s'hl$ cosec ϕ, and its vertical component, obtained by multiplying by sin ϕ is $s'hl$. The total shear on $ACGD$ is $s''hl$. If there is no vertical compression or tension in $CBFG$ or $ABFD$,

$$s''hl = s'hl, \tag{6}$$
$$s'' = s' = s.$$

Whenever a body is subjected to pure shear, the unit shearing stress in planes at right angles to each other is the same.

In many cases, tensile or compressive stresses exist along with the shearing stresses. In Fig. 30, the glue at the left of the middle is in tension, and that at the right of the middle is in compression. All of it is in shear. A portion A of the block is in tension and shear, and a portion C is in compression and shear. The porton B at the middle is in shear only. The direction of the shear (arrows not shown) in A and C is the same as in B.

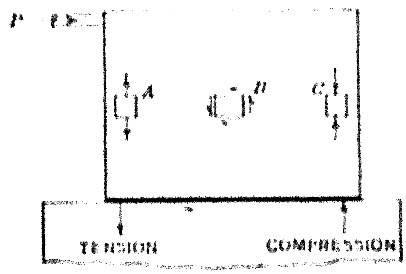


FIG. 30.—Shear with tension and compression.

If the tension at the top and bottom of A is not equal, the vertical shearing stress will not be exactly the same on both sides. Ordinarily, if A is very small, this difference will be slight. The distribution of stress in these cases will be discussed in Chapter X. For the present, we will consider only the cases where Formula III applies.

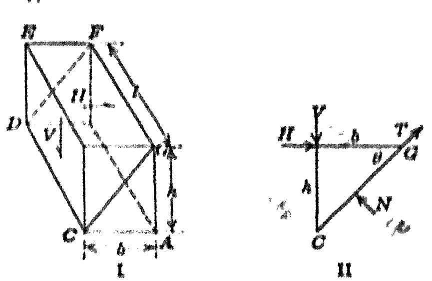


FIG. 31.—Shear causing compression.

31. Compressive and Tensile Stress Caused by Shear.—Fig. 31, I, represents a rectangular parallelopiped of breadth b, height h, and length l, subjected to *pure* shearing stress. The shearing stress acts toward the right parallel to the breadth at the top and toward the left at the bottom. As shown in Article 30, there is also a shearing stress of the same intensity at the left surface acting downward and an equal shearing

stress at the right surface acting upward. (If the direction of one of these shears is reversed, they must all be reversed to produce equilibrium.) Now consider the parallelopiped divided by the inclined plane containing the edges CD and GF, and treat the triangular prism to the left of this plane as a free body in equilibrium under the action of the forces at its surface. These forces are four in number: the shearing force H in the upper surface acting toward the right, the shearing force V in the left vertical surface acting downward, the compressive force N acting normal to the inclined surface (Fig. 31, II, which represents all the forces in the plane of the paper), and a shearing force T along this surface parallel to the diagonal line CG. If s_s is the intensity of the horizontal and vertical shear,

$$H = s_s bl, \quad V = s_s hl.$$

Resolving normal to the inclined plane,

$$N = H \sin \theta + V \cos \theta, \tag{1}$$

$$N = s_s bl \sin \theta + s_s hl \cos \theta, \tag{2}$$

where θ is the angle which the inclined plane makes with the horizontal surface. Since

$$\tan \theta = \frac{h}{b},$$

$$N = 2\, s_s bl \sin \theta. \tag{3}$$

To get the unit compressive stress, s_c, across the inclined surface, divide the total compression N by the area of the surface $bl \sec \theta$:

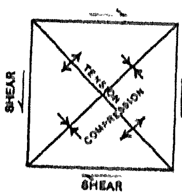


FIG. 32.—Tension, compression and shear.

$$s_c = 2\, s_s \sin \theta \cos \theta = s_s \sin 2\,\theta. \tag{4}$$

When θ is 45 degrees, we get the maximum value of the compressive stress,

$$s_c = s_s. \qquad \textbf{Formula IV.}$$

In like manner, if we consider a second inclined plane perpendicular to CG and parallel to CD, we get a tensile stress of the same value.

When a body is subjected to pure shear, there is a compressive stress of equal intensity across planes at 45 degrees to the direction of the shearing stresses, and tensile stresses of the same intensity across planes at 45 degrees to the shearing planes in the opposite directions. This is shown in Fig. 32.

Problems

1. Prove that a block, subjected to a compressive stress of intensity s and a tensile stress of the same intensity at right angles, has a shearing stress of the same intensity at 45 degrees and 135 degrees (Fig. 33).

2. In Fig. 34, the block ABC is 6 inches long perpendicular to the plane of the paper. Find the unit shearing stress and the unit compressive stress in the glue at the base.

Ans. s_s, 28 pounds per square inch.
s_c, 96 pounds per square inch.

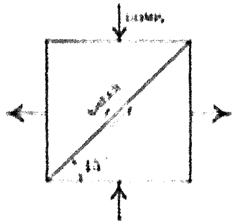


FIG. 33. Shear caused by tension and compression.

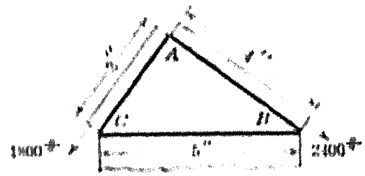


FIG. 34.—Stress due to shear.

3. If ABC, Fig. 34, were an equilateral triangle with each side 5 inches, and the force tangent to each of the inclined faces 1,500 pounds, find the unit shearing and compressive stress at the base.

32. Relation of Shearing to Linear Elasticity. The modulus of shearing elasticity may be calculated from the modulus in tension or compression if Poisson's ratio is known.

Fig. 35 is the front elevation of a block of square section subjected to shearing forces. The unit shearing displacement is

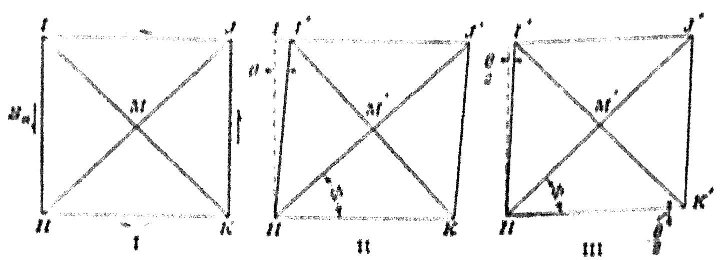


FIG. 35.—Shearing deformation.

the tangent of the angle θ between the lines HI and HI' of Fig. 35, II. In Fig. 35, III, we have rotated the figure an amount equal to one-half of the angle of shear so that the direction of the diagonals remains constant. (This is called *pure shear*, while the case illustrated by Fig. 35, II, is called simple shear.) The amount of this rotation will now be determined.

The shearing forces lengthen the diagonal HJ to HJ' and shorten the diagonal IK to IK'. The half diagonals, HM and MK, suffer the same relative deformation.

If δ is the unit linear deformation due to unit tensile stress s, the unit elongation along HJ is $\delta(1 + \sigma)$. This is made up of the elongation due to the tension along this diagonal and the elongation $\sigma\delta$ due to the equal compressive stress along the other diagonal at right angles to this one. In the same way the unit compression along the diagonal $I'K'$ is $\delta(1 + \sigma)$.

The tangent of the angle ϕ which the line HK' (Fig. 35, III) makes with the diagonal is given by:

$$\tan \phi = \frac{M'K'}{HM'} = \frac{MK[1 - \delta(1 + \sigma)]}{HM[1 + \delta(1 + \sigma)]} = \frac{1 - \delta(1 + \sigma)}{1 + \delta(1 + \sigma)}. \qquad (1)$$

To get $\frac{\theta}{2}$, subtract the angle ϕ from 45 degrees.

$$\text{Tan } \frac{\theta}{2} = \frac{\tan 45° - \tan \phi}{1 + \tan 45° \tan \phi} = \delta(1 + \sigma). \qquad (2)$$

For small angles $\tan \theta = 2 \tan \frac{\theta}{2} = 2\,\delta(1 + \sigma)$. (3)

Since, $\delta = \frac{s}{E}$; $\tan \theta = \frac{2s(1 + \sigma)}{E}$,

$$E_s = \frac{s_s}{\delta_s} = \frac{s_s}{\tan \theta} = \frac{s_s E}{2s(1 + \sigma)}. \qquad (4)$$

But at 45 degrees the tensile and compressive unit stresses are equal to the horizontal and vertical unit shearing stress; $s = s_s$. Canceling these in equation (4),

$$E_s = \frac{E}{2(1 + \sigma)}. \qquad (5)$$

Problems

1. Show that when Poisson's ratio is ¼, $E_s = \frac{2E}{5}$.

2 If E for steel is 30,000,000 pounds per square inch, and Poisson's ratio is 0.27, find the modulus of rigidity.

Ans. 11,800,000 pounds per square inch.

3. Landolt and Börnstein give the following values, in kilograms per square millimeter for cast steel: E = 20,400, E_s = 8,070, E_v = 14,600. Find Poisson's ratio by means of equation (5), and also by means of equation (9), Article 14.

CHAPTER IV

RIVETED JOINTS

33. Kinds of Stress.—Riveted joints afford an excellent illustration of tension, compression, and shear, and of the manner of transmission of stress. Fig. 36 represents a pair of plates, each of breadth b and thickness t, transmitting a pull P in the direction of their length. The plates are united by means of a pin C, which fits tightly in a hole in the lower plate and passes through a hole in the upper plate. If we consider the upper plate, we find that the portion to the left of the pin is in tension.

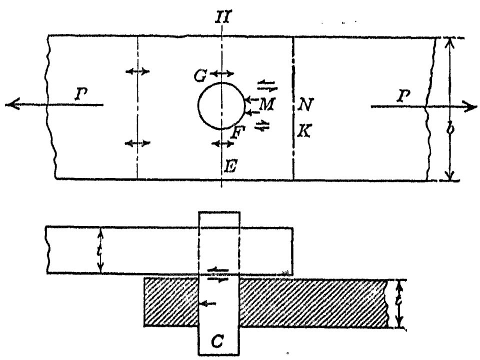


Fig. 36.—Stress at a bolted joint.

The intensity of this tensile stress is found by dividing the pull P by the area bt. At the section EH in the plane of the center of the hole, the stress is still tension. The unit stress is greater here, for the area of cross-section is diminished by the material cut away to make room for the pin. If the hole is in the middle of the section and in the line of the pull, half of the total stress is transmitted by the lower section EF and half by the upper section GH. The total stress which passes EF as tension passes FK as shear. The intensity of this shearing stress in the plate may be calculated by dividing the pull, $\frac{P}{2}$, by the section of length FK and thickness t. At M, the surface

of contact of the pin and plate, the stress is compression. The force is transmitted as a shearing stress from the part of the pin in the upper plate to the portion in the lower plate, and finally as compression to the lower plate.

It helps to fix our ideas if we regard stress as flowing like an electric current. This is illustrated in Fig. 37. We may regard the circuit as completed through the bodies which exert the pull on the plates.

In calculating the unit shearing stress in the plates behind the bolt or pin, since there is some uncertainty as to the width of the bearing surface at M, Fig. 36, it is customary to take the distance MN instead of FK in getting the shear area.

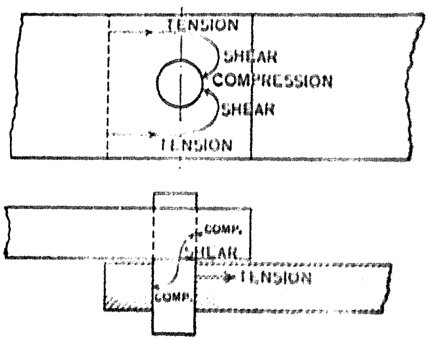


FIG. 37.—Flow of stress.

Problems

1. The plates in Fig. 37 are each 3 inches wide and ½ inch thick. The hole in the upper plate is 1 inch in diameter and that in the lower plate is ⅞ inch in diameter. The bolt is ⅞ inch in diameter and the total pull is 5,100 pounds. Find the unit shearing stress in the bolt and the unit tensile stress in the net section of each plate.

Ans. s_s, 8,480 pounds per square inch.
s_t, 5,100 pounds per square inch, net section, upper plate.
s_t, 4,800 pounds per square inch, net section, lower plate.

2. In Problem 1 find the unit shearing stress in the upper plate to right of the bolt if the center of the hole is 1.5 inches from the right edge of the plate.

Ans. 5,100 pounds per square inch.

34. Bearing or Compressive Stress.—In calculating the unit bearing or compressive stress at the surface of contact of the pin and plate, it is customary, among engineers, to regard the bearing area as the product of the thickness of the plate multiplied

Bearing means compressive

by the diameter of the pin. If d is the diameter of the pin and t is the thickness of the plate, the *bearing area* is td. In other words, it is the projection upon a plane parallel to the axis of the pin of that portion of the pin which is inside of the plate. Consider Fig. 38, in which a rectangular bar of thickness d is placed across the edge of a plate of thickness t. If the bar crosses the plate at right angles, it is plain that the area of contact is td. If, as in Fig. 39, the bar passes through a hole in the plate, the bearing area is the same; and if the forces P_1, P_2 are balanced with respect to the center of the plate, the bearing stress is uniform over the entire area. If the forces are not balanced, the area remains the same and the average bearing stress is the same, but the maximum stress is greater. If there is force on only one

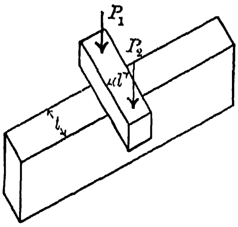


FIG. 38.—Bearing.

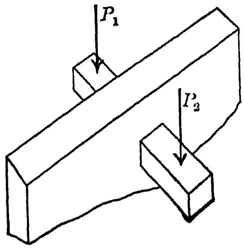


FIG. 39.—Bearing.

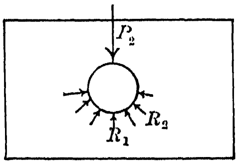


FIG. 40.—Bearing.

side of the plate, the smaller the distance between the force and plate the smaller will be the maximum bearing stress. In Fig. 40 we have a round pin or bolt passing through a plate. The actual area is the lower half of the surface of the cylinder, of length t and diameter d. The reactions R_1, R_2, etc., are not all vertical, but are nearly normal to the surface of contact. If, as in the case of liquid pressure, these reactions were exactly normal and of equal intensity, the resultant of their vertical components would be the same as if that unit pressure were exerted on the horizontal projection of this cylindrical surface.

Problems

1. In Problem 1 of Art. 33 what is the unit compressive stress of the pin? *Ans.* 11,657 pounds per square inch.

2. In Fig. 36 the diameter of the bolt and of the hole in the lower plate is 7/8 inch. What must be the thickness and width of the plate in order

that the unit bearing stress shall be 15,000 and the unit tensile stress in the net section shall be 12,000 pounds per square inch when the unit shearing stress in the bolt is 7,500 pounds per square inch?

Ans. Thickness, 0.34 inch.
Width, 2 inches, nearly.

35. Lap Joint with Single Row of Rivets.—Fig. 41 shows a *lap joint* with a single row of rivets. In any riveted joint the distance p from center to center of adjacent rivets in a row is called the *pitch*. In solving problems, it is often convenient to consider a single strip of width p alone. In this case the problem of a lap joint with a single row of rivets becomes the same as that of Article 33. We may take this strip as extending from center to center of adjacent rivets, as in the lower part of Fig. 41. In this case, the total tension is transmitted in the plate between the two rivets, and the shear is equally divided between the upper half of the lower rivet and the lower half of the upper rivet. Or we may take the strip as including a single rivet, as in the upper portion of Fig. 41, in which case the shear is transmitted by a single rivet and the tension is divided.

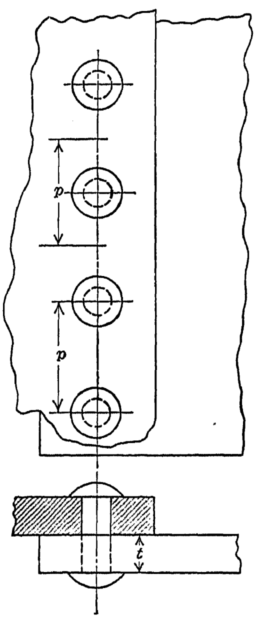


FIG. 41.—Single-riveted lap joint.

In problems in riveting, unless otherwise stated we shall consider the rivet as exactly filling the rivet hole, and that the holes are drilled or reamed so that there is no injured material around them due to overstrain while punching. In practice, where the holes are punched and not reamed it is customary to make some allowance for this injured material.

In problems where the width of the plate is given, it is generally better to consider the entire plate as a unit. In Problem 1 below, to get the tensile stress in the net section we may take a strip 8.25 inches wide transmitting a pull of 13,200 pounds.

Problems

(Look up rivet areas in handbook)

1. Two ½-inch plates, each 12 inches wide, are united by five ¾-inch rivets in a single row to form a lap joint. The joint transmits a pull of 13,200 pounds. Find the unit tensile stress in the gross section of the plates,

the unit tensile stress in the net section, the unit shearing stress in the rivets, and the unit compressive stress between rivets and plates.

Ans. $\begin{cases} s_t, \text{ 3,200 pounds per square inch in net section.} \\ s_s, \text{ 5,975 pounds per square inch.} \\ s_c, \text{ 7,040 pounds per square inch.} \end{cases}$

2. Two $\frac{3}{8}$ inch boiler plates are united by a single row of $\frac{15}{16}$-inch rivets to form a lap joint. The pitch is $2\frac{3}{8}$ inches. Find the unit tensile stress in the net section when the unit shearing stress in the rivets is 8,800 pounds per square inch. *Ans.* 11,260 pounds per square inch.

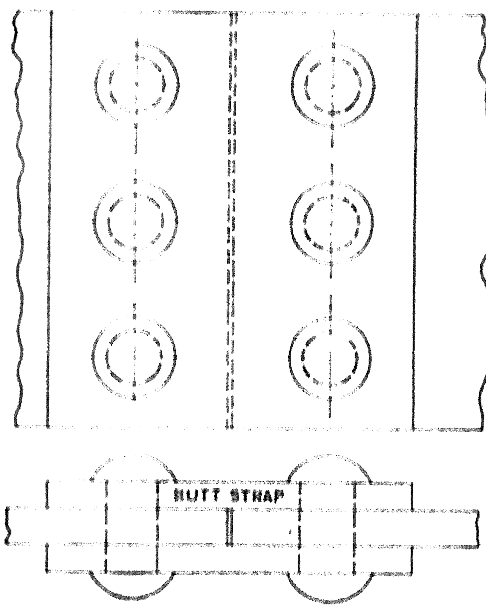


FIG. 42.—Single-riveted butt joint.

36. Butt Joint.—A butt joint is made when the two principal plates are in the same plane, and are united by means of one or two additional narrow plates called *cover plates* or *butt straps*. A butt joint with a single butt strap is equivalent to two lap joints placed tandem.

Fig. 42 shows a butt joint with double butt straps. The rivets are in double shear so that the total shear transmitted by each rivet is twice as great as in a lap joint.

Example

Two $\frac{1}{2}$-inch plates are united to form a butt joint by two $\frac{5}{16}$-inch butt straps. There is one row of $\frac{7}{8}$-inch rivets on each side. If the allowable

unit tensile stress in the plate is 10,000 pounds per square inch and the allowable unit shearing stress in the rivets is 8,000 pounds per square inch, what should be the pitch?

The area of one rivet is 0.6013 square inch, and each rivet is in double shear. The net cross-section which carries the tension equal to the shear in one rivet is $\frac{1}{2}\left(p - \frac{7}{8}\right)$ so that,

$$\frac{1}{2}\left(p - \frac{7}{8}\right) 10{,}000 = 2 \times 0.6013 \times 8{,}000,$$

$$p - \frac{7}{8} = 1.924 \text{ inches},$$

$$p = 2.80 \text{ inches}.$$

Problems

1. Two ⅜-inch plates are united by two ¼-inch butt straps to form a butt joint. There is one row of ¾-inch rivets on each side. The pitch is 2½ inches. Find the unit tensile stress in the net section of the ⅜-inch plates and the unit bearing stress between these plates and the rivets when the unit shearing stress in the rivets is 6,000 pounds per square inch.

Ans. $s_t = 8{,}079$ pounds per square inch.
$s_c = 18{,}850$ pounds per square inch.

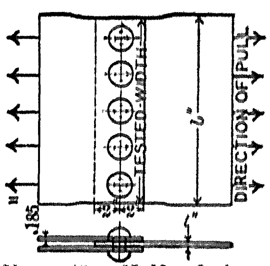


FIG. 43.—Half of butt joint.

Fig. 43 represents a set of tests made at the Watertown Arsenal, to study the behavior of riveted joints. A plate of width b and thickness t was planed down for a portion of its length to some convenient width and united to a pair of plates, thus forming one-half of a butt joint. Wrought-iron rivets were used of nominal diameter 1/16 inch less than the diameter of the holes. In calculating it was assumed that the finished rivets entirely filled the rivet holes.

Problems

2. In test piece No. 1,353 (Watertown Arsenal, 1885, page 867), the breadth b was 14.90 inches; the tested width, 14.39 inches; the actual thickness of the plate, 0.248 inch. There were five rivets in 1-inch drilled holes. The joint failed by tension along the line of the rivet holes under a pull of 156,440 pounds. The calculated results as published are:

AREAS	Square inches
Gross sectional area of plate.	3.569
Net sectional area of plate.	2.329
Bearing surface of rivets.	1.240
Shearing area of rivets.	7.854

MAXIMUM STRESS ON JOINT	Pounds per square inch
Tension in gross section of plate..........	43,830
Tension in net section of plate...........	67,170
Compression on bearing surface of rivets...	126,160
Shearing in rivets	19,920

Verify these results.

3. In test piece No. 1,355 the results were:

Tested width of plate.........................	15 inches.
Actual thickness............................	0.251 inch.
Ultimate load.	167,200 pounds.

There were five rivets in 1-inch holes. "Fractured two outside sections of plate at edge along line of riveting; the two middle sections sheared in front of the rivets."

Compute all unit stresses as in Problem 2.

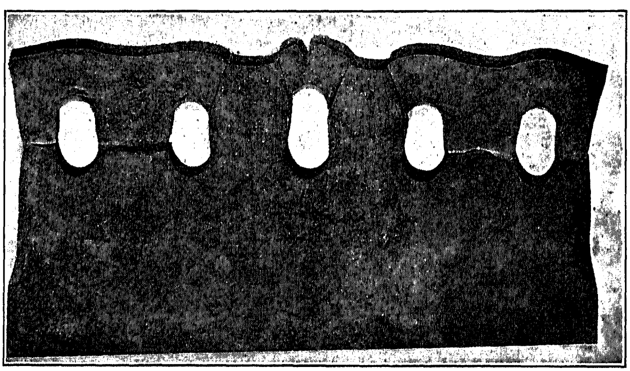


FIG. 44.—Failure of riveted plate.

Fig. 44 is a copy of a photograph of this plate after failure. It shows failure by tension across the net section and shear in front of the rivets. It also shows elongation of the rivet holes due to bearing pressure on the plate, combined with shear.

In order to compare the strength of the material in the net section of a riveted joint with the ordinary tension tests, two strips were sheared from each sheet of steel, one lengthwise, the other crosswise the sheet. These were planed to a width of 1.5 inches and tested in the usual way.

From the sheet used in No. 1,353 two test pieces were taken. These gave as ultimate tensile strengths:

	Pounds per square inch
No. 1,213, lengthwise.......................	59,180
1,224, crosswise.........................	60,840

Four test strips were taken from the sheet used for No. 1,355:

	Pounds per square inch
No. 1,214, lengthwise	58,680
1,220, lengthwise	62,300
1,225, crosswise	61,230
1,226, crosswise	60,890

Comparing these results with the unit stresses in the net section of the riveted plates, we find that the stress in the test pieces is considerably lower. This is an illustration of Article 24. The net section in the riveted plate is relatively short and consequently is kept from necking as it would in a longer piece. Notice that there is no certain difference between the pieces lengthwise and those crosswise the plate. This is explained by the fact that in rolling the metal it was worked both ways, so that there was no definite grain in one direction.

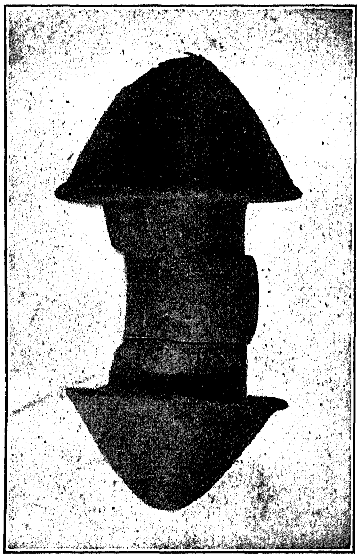


FIG. 45.—Failure of rivet.

In Problems 2 and 3, the design was such that there was relatively small shearing stress. The rivets used were large compared with the thickness of the plate. Problem 4, below, represents a different case with a different mode of failure.

Problems

4. In a test piece similar to Fig. 43 (Watertown Arsenal, 1886, page 1401), the following data are given; tested width, 13.11 inches; thickness, 0.630 inch; five rivets in 1-inch drilled holes; failed by shearing the rivets under a pull of 295,500 pounds; rivet holes elongated 0.31 inch, 0.32 inch, 0.26 inch, 0.25 inch, 0.24 inch.

Calculate the unit stresses.

		Pounds per square inch
Ans.	Tensile stress in net section	57,840
	Bearing stress	93,810
	Shearing stress on rivets	37,620

5. In Problem 4, the butt straps were 0.384 inch thick. Find the unit tensile stress in the net section.

Fig. 45 is a copy of a photograph of a rivet which failed by shear as in Problem 4 (Watertown Arsenal, "Tests of Metals," 1886, page 1567).

37. Rivets in More Than One Row.—Rivets are frequently arranged in two or more rows. The rivets in the second row may be placed directly behind the rivets in the first row, or they may be arranged zigzag as shown in Fig. 46. Two adjoining rows of zigzag rivets must not be placed too close together or the plate will fail along the diagonal line joining the rivets of the two rows. The Boiler Code of the American Society of Me-

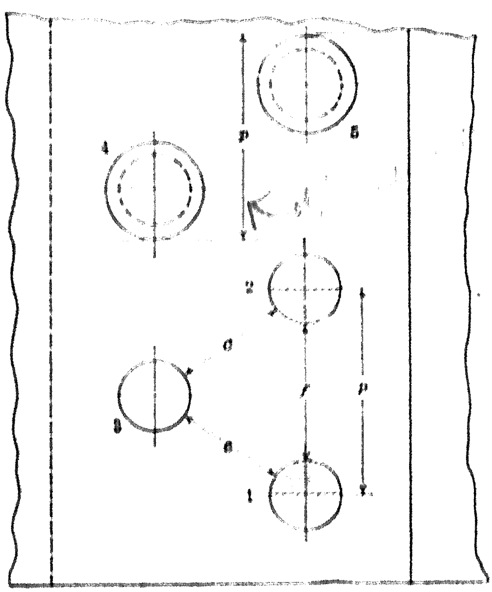


FIG. 46.—Double-riveted lap joint.

chanical Engineers specifies that the total net distances, *e* and *e* of Fig. 46, must exceed the net distance *f* by at least 20 per cent.*

In computing problems of two or more rows of rivets it is customary to regard the unit shearing stress the same in all rivets.

Where narrow plates are united by several rows of rivets, it is best to take the entire width of the plate as the unit.

* *Transactions of the American Society of Mechanical Engineers*, 1914, page 1018.

Example

Two ¾-inch by 12-inch plates are united to form a lap joint by means of ten 1-inch rivets arranged as shown in Fig. 47. The joint transmits a pull of 60,000 pounds. Find the unit shearing stress in the rivets and the unit tensile stress in plate A at sections 1–1, 2–2 and 3–3.

$$s_s = \frac{60{,}000}{10 \times 0.7854} = 7{,}639 \text{ pounds per square inch.}$$

At section 1–1 the net area is $(12 - 1)\frac{3}{4}$ and

$$s_t = \frac{60{,}000}{11 \times \frac{3}{4}} = 7{,}273 \text{ pounds per square inch.}$$

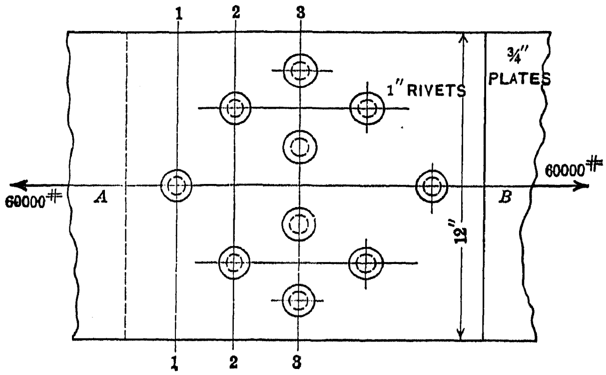


FIG. 47.—Multiple-riveted lap joint.

At section 2–2 the net width is 10 inches, but since one-tenth of the total pull has been transmitted by rivet 1 from plate A to plate B the total tension transmitted through this net section is only 54,000 pounds, and

$$s_t = \frac{54{,}000}{10 \times \frac{3}{4}} = 7{,}200 \text{ pounds per square inch.}$$

At section 3–3 the net width is 8 inches, but three-tenths of the total pull has been transmitted to plate B through the rivets in sections 1–1 and 2–2 so that the total tension in net section 3–3 is only 42,000 pounds.

$$s_t = \frac{42{,}000}{6} = 7{,}000 \text{ pounds per square inch.}$$

Problem

1. Solve the above example if the plates are 10 inches wide instead of 12 inches.

$$\textit{Ans.} \begin{cases} \text{At 1–1, } s_t = 8{,}889 \text{ lb./in.}^2 \\ \text{At 2–2, } s_t = 9{,}000 \text{ lb./in.}^2 \\ \text{At 3–3, } s_t = 9{,}333 \text{ lb./in.}^2 \end{cases}$$

Notice that in problem 1 the greatest tensile stress is at section 3–3, while in the example the greatest stress is at section 1–1.

In wide plates, such as are used in boilers, it is not convenient to consider the entire width, but it is better to divide the width up into a number of equal units, each of which includes a group of rivets. Generally the width of such a unit is the pitch in the row having the fewest rivets.

In Fig. 46, where the pitch of both rows is the same, the width of the unit is equal to the pitch. It may extend from center to center of two rivets in one row. The entire pull which is transmitted by a strip of the plate of width p may be regarded as carried by the whole of rivet 3, the shaded portion of rivet 2, and the shaded portion of rivet 1; or a strip of equal width may include all of rivets 4 and 5. In either case, one rivet in each row is included in the unit. All the stress which is transmitted by a strip of unit width passes through the net section of the lower plate between two rivets of the right row. Half of this stress is transmitted to the upper plate through the rivet of the right row, and the other half passes through the net section at the left row of rivets. The unit tensile stress in the net section at the right row in the lower plate and at the left row in the upper plate is twice as great as the unit stress in the left row in the lower plate and the right row in the upper plate.

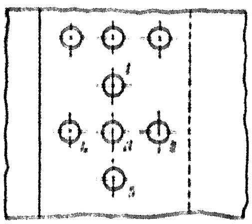


FIG. 48.—Rivets in three rows.

Fig. 48 shows three rows of rivets with twice as many rivets in the middle row as in either of the others. The unit strip is taken as equal to the pitch in the outer rows. The unit strip may extend from the center of rivet 1 to the center of rivet 5, or it may include the whole of rivet 1 and none of rivet 5. In either case it embraces two rivets in the middle row and one rivet in each of the others.

Problems

2. Two $\frac{3}{8}$-inch plates are united by $\frac{3}{4}$-inch rivets to form a lap joint. The rivets are in three rows as in Fig. 48 with the pitch in the outer rows 5 inches. Find all the unit stresses if the gross section of the plates transmits a pull of 2,400 pounds per inch of width.

Ans.	Pounds per square inch
s_s in all rivets	6,790
s_c	10,667
s_t right upper and left lower	7,530
s_t left upper and right lower	1,882
s_t at middle rows	6,857

3. A butt joint is formed of two $\frac{1}{2}$-inch plates united by two $\frac{5}{16}$-inch butt straps. There are two rows of $\frac{7}{8}$-inch rivets on each side, the inner rows of 3-inch pitch and the outer rows of 6-inch pitch. The unit stress in the gross section of the $\frac{1}{2}$-inch plates is 6,000 pounds per square inch. Find the unit tensile stress in the net section at each row of rivets in the $\frac{1}{2}$-inch plates and at one of the inner rows in the butt straps. Find the unit shearing stress in the rivets and the unit bearing stress between the rivets and the $\frac{1}{2}$-inch plates.

Ans.	Pounds per square inch
s_t in $\frac{1}{2}$-inch plates in outer rows	7,024
s_t in $\frac{1}{2}$-inch plates in inner rows	5,647
s_t in butt straps at inner rows	6,776
s_s in all rivets	4,989
s_c between rivets and main plates	13,714

4. Solve Problem 3 if one of the butt straps is narrow so as to include only the two inner rows of rivets while the other takes in all the rivets, Fig. 49.

Ans.	Pounds per square inch
s_t in $\frac{1}{2}$-inch plates at outer rows	7,024
s_t in $\frac{1}{2}$-inch plates at inner rows	6,776
s_t in narrow butt strap at inner rows	5,421
s_t in wide butt strap at inner rows	8,132
s_s in all rivet sections	5,987
s_c in outer rows	8,229
s_c in inner rows	16,457

Suggestion.—The outer rivet being in single shear and the two inner rivets being in double shear, one-fifth of the total pull on the 6-inch strip is transmitted to the wide butt strap at the outer row.

5. What should be the minimum thickness of the wide butt strap of Problem 4 in order that the unit stress in it may not exceed the maximum stress in the $\frac{1}{2}$-inch plates?

Joints with one narrow and one wide butt strap are considerably used in boiler work. Such joints are open to the objection that the pull is not equal on both sides so that some bending is produced in the plates, which decidedly weakens the joint. In a lap joint the effect of bending stresses is still greater, and such joints should be designed for much lower unit tensile stresses than are allowable in a butt joint with equal butt plates.*

* F. W. Dean, *Transactions of the American Society of Mechanical Engineers*, 1915, page 620.

38. Efficiency of a Riveted Joint.—The ratio of the strength of a riveted joint to the strength of one of the plates which it unites is called the *efficiency* of the joint. The efficiency may also be defined as the ratio of the unit stress in the gross section, when the joint is stressed to its allowed limit, to the allowable unit stress in the plates. If the joint is so designed as to make it at least as strong in shear and compression as it is in tension at the net section, the efficiency becomes the ratio of the net section to the gross section.

The calculations for efficiency may be based upon either the ultimate strength or the allowable unit stress. The Boiler Code of The American Society of Mechanical Engineers specifies:

	Pounds per square inch
Tensile strength of steel plate	55,000
Crushing strength of steel plate	95,000
Shearing strength of steel rivets	44,000
Shearing strength of iron rivets	38,000

The code further specifies that the strength of a rivet in double shear is twice its strength in single shear.

With a factor of safety of 5, which is the one generally used in boiler design, the allowable unit stresses become:

	Pounds per square inch
s_t for steel	11,000
s_c for steel	19,000
s_s for steel	8,800
s_s for iron	7,600

To find the efficiency of a given joint calculate the strength of a unit width in tension at the net sections, in shear at the rivets, and in bearing between rivets and plates, and divide the smallest of these by the tensile strength of the unit width of the plate.

Example

Two 3/8-inch steel plates are united by a double row of 3/4-inch steel rivets to form a lap joint. The pitch in each row is 2 3/4 inches. Using the ultimate strengths of the A. S. M. E. Boiler Code,* calculate the efficiency of the joint.

The unit strip is 2 3/4 inches wide, and the net width between rivets is 2 inches.

* *Transactions of the American Society of Mechanical Engineers*, 1914

Tensile strength of net section	$= 2 \times \frac{3}{8} \times 55{,}000 = 41{,}250$ pounds
Shearing strength of rivets	$= 2 \times 0.4418 \times 44{,}000 = 38{,}878$ pounds.
Bearing strength	$= 2 \times \frac{3}{8} \times \frac{3}{4} \times 95{,}000 = 53{,}437$ pounds
Tensile strength of unit	$= 1\frac{1}{4} \times \frac{3}{8} \times 55{,}000 = 56{,}719$ pounds

The joint is weakest in shear and

$$\text{Efficiency} = \frac{38{,}878}{56{,}719} = 0.685 = 68.5 \text{ per cent.}$$

Problems

1. Two $\frac{3}{8}$-inch steel plates are united by a double row of $\frac{7}{8}$-inch steel rivets to form a lap joint. The pitch is $3\frac{3}{8}$ inches. Find the efficiency of the joint. *Ans.* Weakest in tension; efficiency = 74.1 per cent.

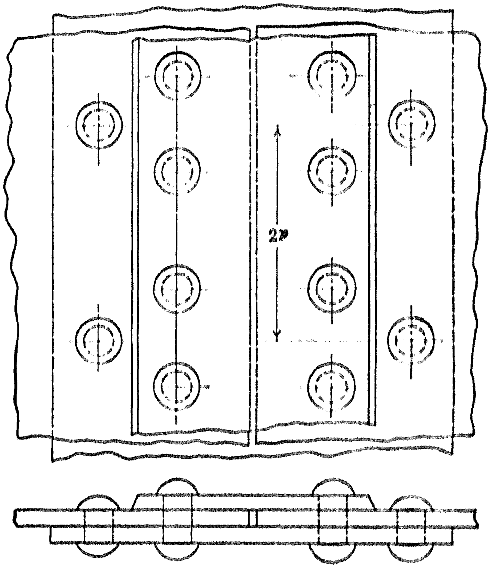


FIG. 49.—Double-riveted butt joint.

2. Two $\frac{1}{2}$-inch plates are united to form a butt joint. There are two rows of $\frac{3}{4}$-inch rivets on each side. The pitch at the inner rows is 2.5 inches and at the outer rows it is 5 inches. Find the efficiency.

Ans. Weakest in compression; efficiency is 77.7 per cent.

3. What should be the minimum thickness of the butt straps in Problem 2 in order that they may be 10 per cent. stronger than the main plates at their weakest section.

Example

In Fig. 49 the pitch of the outer row is 5 inches, the pitch of the inner row is 2.5 inches, the diameter of the rivets is $\frac{3}{4}$ inch, and the thickness of the plates is $\frac{3}{8}$ inch; find the strength and efficiency of the joint.

Taking a strip 5 inches wide as a unit there are two rivets in double shear and one rivet in single shear.

	Pounds
Shearing strength of one rivet section = 0.4418 × 44,000	19,439
Bearing strength of one rivet = 3/8 × 3/4 × 95,000	26,718
Total shearing strength 5 × 19,439	97,195
Tension in outer net section = 3/8 × 4¼ × 55,000	87,656
Tension in inner net section = 3/8 × 3½ × 55,000	72,187
Tension in gross section = 3/8 × 5 × 55,000	103,125

The joint may fail by tension at inner section with shear or compression at outer rivet.

Tension at inner section and shear = 72,187 + 19,439	91,626

The joint may fail by compression at the two inner rivets and shear at the outer rivet.

26,718 × 2 + 19,439	72,875

As the compressive strength of the outer rivet is greater than the shearing strength it was not necessary to calculate the total strength in compression or the effect of tension at inner section with compression in outer rivet.

$$\text{Efficiency} = \frac{72{,}875}{103{,}125} = 70.7 \text{ per cent.}$$

Problems

4. Solve the example above using 7/8-inch rivets.

Ans. Weakest in tension in the outer net section; efficiency, 82.5 per cent.

5. Fig. 50 shows one of a series of quadruple-riveted butt joints designed by the Hartford Boiler Inspection and Insurance Co. The short pitch is 4⅛ inches and the rivet holes are 1 3/16 inch. Find the efficiency, using the unit stresses of the A. S. M. E. Boiler Code.

Ans. 92.7 per cent. Joint fails by tearing the plate between rivet holes in the third row and shearing the rivets in the two outer rows.

39. Joints of Maximum Efficiency.—It is possible to design a joint having the same strength in tension, compression, and shear. Such designs frequently lead to unusual sizes of rivets or to values of pitch which will not leave room for forming the rivet head. In structural work the pitch is required to be at least three times the diameter of the rivet. Smaller pitch is permitted in boilers.

Formulas for joints of maximum efficiency are useful in deriving approximate values of the pitch and sizes of rivets, and in showing the maximum efficiency possible with a given style of joint.

Single-riveted Lap Joint.—Equating the compressive strength with the tensile strength in the net section,

$$s_c td = s_t t(p - d), \tag{1}$$

where t is the thickness of the plate, d is the diameter of the

rivet, p is the pitch, and s_c and s_t are the compressive and tensile stress.

$$s_c d = s_t(p - d), \tag{2}$$

$$p = \frac{s_c + s_t}{s_t} d. \tag{3}$$

$$\text{Efficiency} = \frac{p - d}{p} = 1 - \frac{d}{p} = 1 - \frac{s_t}{s_c + s_t} = \frac{s_c}{s_c + s_t}. \tag{4}$$

no. of shear = 19 = m
no. of crushing = 11 = n

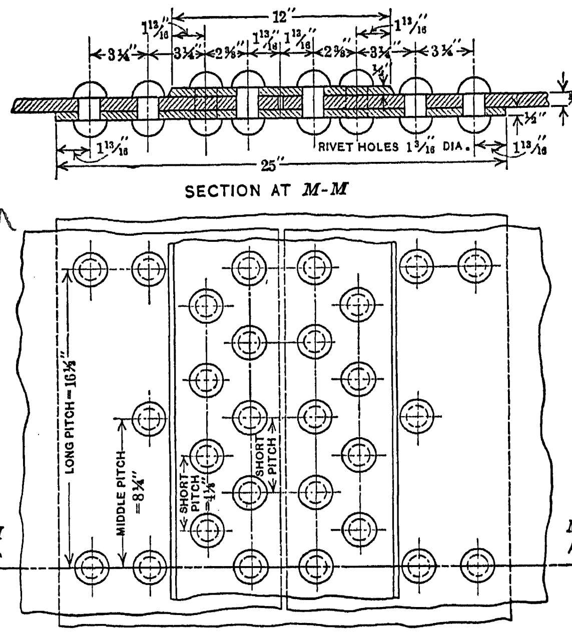


FIG. 50.—Quadruple-riveted butt joint.

Equating the shearing strength and the compressive strength,

$$s_c t d = \frac{s_s \pi d^2}{4}, \tag{5}$$

$$d = \frac{4\, s_c t}{\pi s_s}. \tag{6}$$

If $s_c = 95{,}000$, $s_s = 44{,}000$ and $s_t = 55{,}000$,

$$d = \frac{4 \times 95}{44\,\pi} t = \frac{95}{11\,\pi} t = 2.75\, t \text{ nearly.}$$

$$p = \frac{150}{55} d = \frac{30}{11} d = 7.5\, t \text{ nearly.}$$

$$\text{Efficiency} = \frac{95}{150} = 0.633 = 63.3 \text{ per cent.}$$

Problem.

1. A single-riveted lap joint is made of ¼-inch steel plates. Find the diameter and pitch of the rivets for maximum efficiency.

Ans. $d = {}^{11}\!/_{16}$-inch; $p = 1\frac{7}{8}$ inches.

The pitch in Problem 1 is less than the allowable value for structural-steel work but is permissible in boiler work.

Single-riveted Butt Joint.—In a single-riveted butt joint the relations between compression of rivets and tension in the net section is the same as in single-riveted lap joints, so that equations (1), (2), (3), and (4) are valid.

Since each rivet is in double shear, (5) becomes

$$s_c td = \frac{2\, s_s \pi d^2}{4}, \tag{7}$$

$$d = \frac{2\, s_c t}{\pi s_s}.$$

When s_c is 95,000, s_s is 44,000 and s_t is 55,000 pounds per square inch,

$$d = 1.374\, t = \frac{11}{8}\, t \text{ nearly.} \tag{8}$$

$$p = \frac{150}{55} \times \frac{11}{8}\, t = 3.75\, t.$$

Problems

2. Two ⅝ nch plates are united to form a single-riv ted butt joint. Find the size of rivets and the pitch for maximum efficiency.

Ans. $d = 0.86$ inch; $p = 2.35$ inches.

Use $d = 0.875 = \frac{7}{8}$ inch; $p = 2.375 = 2\frac{3}{8}$ inches.

3. In problem 2, as used, find the efficiency.

Ans. Weakest in tension in net section; efficiency = 63.2 per cent.

Double-riveted Lap Joint.—In a double-riveted lap joint there are two rivets in compression in unit width and

$$2\, s_c td = s_t t(p - d), \tag{9}$$

$$p = \frac{2\, s_c + s_t}{s_t} d. \tag{10}$$

$$\text{Efficiency} = \frac{2\, s_c}{2\, s_c + s_t}. \tag{11}$$

The relation of shear and compression is the same as in a single-riveted lap joint so that equation (6) is valid.
If $s_c = 95{,}000$, $s_t = 55{,}000$, and $s_c = 44{,}000$,

$$p = \frac{49}{11} d; \quad d = \frac{11}{4} t.$$

$$\text{Efficiency} = \frac{38}{49} = 0.775 = 77.5 \text{ per cent.}$$

Problems

4. Two $\frac{5}{16}$-inch plates are united to form a double-riveted lap joint. Find the size of the rivets and the pitch for maximum efficiency.
Ans. $d = 0.86$ inch; $p = 3.83$ inches.
Use $d = \frac{7}{8}$ inch; $p = 3\frac{7}{8}$ inches.

5. Find the efficiency of the joint of Problem 4 with d and p as used.
Ans. Weakest in tension; efficiency = 77.4 per cent.

6. Two $\frac{3}{4}$-inch plates are united by a double row of rivets to form a lap joint. Find the size of rivets and the pitch for maximum efficiency.
Ans. $d = 2.06$ inches, an impractical rivet.

Double-riveted Butt Joint.—If all rivets pass through both butt straps the relation of tension and compression is the same as in a double-riveted lap joint and equations (10) and (11) are valid. Equations (7) and (8) are valid for all butt joints in which all rivets are in double shear. One advantage of butt joints over lap joints is the fact that smaller rivets may be used with thick plates and still have the maximum efficiency.

Problem

7. Two $\frac{5}{8}$-inch plates are united to form a double-riveted butt joint in which all rivets are in double shear. Find the diameter of rivets and the pitch for the maximum efficiency.
Ans. $d = 0.86$; $p = 3.83$ inches.
Use $d = \frac{7}{8}$; $p = 3\frac{7}{8}$ inches.

Double-riveted Butt Joint with Outer Row Twice the Pitch of Inner Row.—There are three rivets in the unit width. If p is the pitch at the outer row

$$3\, s_c d = s_t (p - d), \tag{12}$$

$$p = \frac{3\, s_c + s_t}{s_t}. \tag{13}$$

$$\text{Efficiency} = \frac{3\, s_c}{3\, s_c + s_t} \tag{14}$$

Equations (7) and (8) apply.

If $s_c = 95{,}000$, $s_t = 55{,}000$, and $s_s = 44{,}000$,

$$d = \frac{11}{8}t;$$

$$p = \frac{340}{55}d = \frac{68}{11}d = 8.5t.$$

$$\text{Efficiency} = \frac{285}{340} = 83.8 \text{ per cent.}$$

Problem

8. Two ½-inch plates are united with double butt straps of equal width. There are two rows of rivets on each side, the pitch of the rivets in the outer rows being twice that of the inner rows. Find the diameter and pitch for the maximum efficiency. *Ans.* $d = 1\frac{1}{16}$ inch; $p = 4\frac{1}{4}$ inches.

These principles may be applied to any arrangement of rivets, but care must be taken to see that the joint is not weaker by some combination of stresses than it is by a single stress. This was illustrated by the last example of the preceding article.

It is necessary that boilers should be tight as well as strong, and for this reason rivets are frequently used in the construction of butt joints somewhat too large for the maximum efficiency.

The Boiler Committee of the American Society of Mechanical Engineers has worked out the efficiency of a number of joints. These are found in the *Transactions* of 1914, pages 1067 to 1075.

The design of riveted joints for steel construction is treated fully in treatises on the "Design of Framed Structures."

40. Stresses in Hollow Cylinders and Spheres.—In a hollow vessel inclosing a liquid or gas under pressure, the pressure of the fluid develops stresses in the walls of the vessel. The pressure of a fluid at any point is normal to the surface. The resultant pressure on any portion of a curved surface in any *given direction* is equal to the pressure on a plane surface perpendicular to the given direction and equal in area to the projection of the curved surface upon its plane. Fig. 51 represents a portion of the surface of a cylinder of diameter D, and of length l perpendicular to the plane of the paper. If P is the pressure on this surface in pounds per square inch, the total pressure on the semicircular surface to the right of the plane AB is $\frac{P\pi Dl}{2}$. The re-

sultant pressure on this surface in the direction normal to AB is PDl, since Dl is the area of the projection of the curved surface upon the vertical plane. There is an equal pressure in the opposite direction upon the curved surface to the left of AB. These two equal and opposite forces are resisted by the circumferential tensile stresses in the sections at A and B. If t is the thickness of the wall of the cylinder, the area in tension is $2\,tl$, and

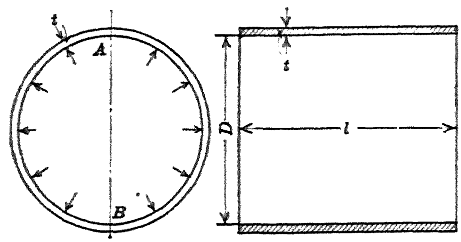


FIG. 51.—Hollow cylinder with internal pressure.

$$2tls_t = PDl, \qquad (1)$$

$$s_t = \frac{PD}{2\,t}. \qquad \text{Formula V.}$$

Problems

1. A boiler shell 30 inches in diameter is subjected to a steam pressure of 150 pounds per square inch. The plates are $\frac{3}{8}$ inch thick. Find the unit stress. *Ans.* 6,000 pounds per square inch.

2. A boiler shell 5 feet in diameter is made of $\frac{9}{16}$-inch plates. The longitudinal joints have an efficiency of 80 per cent. The allowable unit tensile stress in the plates is 11,000 pounds per square inch. Find the allowable pressure. *Ans.* 165 pounds per square inch.

The resultant pressure on the end of a hollow cylinder in the direction of its length is the product of the pressure in pounds per square inch multiplied by the area of cross-section of the cylinder. This is resisted by a longitudinal tensile stress in the shell. The cross-section in tension is approximately the inner circumference multiplied by the thickness.

$$s_t \pi Dt = \frac{P\pi D^2}{4}, \qquad (3)$$

$$s_t = \frac{PD}{4\,t}. \qquad (4)$$

The longitudinal unit stress in a hollow cylinder is one-half the circumferential unit stress.

Equations (3) and (4) apply also to hollow spheres subjected to internal pressure.

Since the longitudinal tensile stress is only one-half the circumferential unit stress, it is only necessary to have the efficiency of the circumferential joints which resist the longitudinal tension a little greater than one-half the efficiency of the longitudinal joints.

Problem

3. A hollow cylinder 4 feet in diameter is made of 5/16-inch plates. The longitudinal joints are double-riveted lap joints using 7/8-inch rivets spaced 3 7/8 inches (Problem 4 of Article 39). The circumferential joints are single-riveted lap joints with 7/8-inch rivets spaced 2 1/2 inches. Find the allowable pressure using A. S. M. E. ultimate strengths with a factor of safety of 5.

If there is a [illegible]

then

$$PD = 2tS \times eff.$$

for instance if it is 77% efficiency

$$\left(PD = 2tS_t \times \frac{77}{100} \right).$$

See note-book for extra problems.

CHAPTER V

TORSION

41. Torque.—A shaft or rod subjected to a pair of equal and opposite couples in parallel planes at right angles to its length is in torsion between these planes. In Fig. 52 we have a rope wound around a shaft and carrying a weight. Attached to the shaft is a pulley upon which runs a belt. The tension on the rope and the reactions of the bearings form a counterclockwise couple, while the tension on the belt and the reactions form a clockwise couple. If there is no friction at the bearings, these couples are equal, provided the shaft is stationary or moving in either direction with constant speed. The moment of either couple is the *twisting moment*, or *torque*, in the portion of the shaft between the pulley and the rope. We will represent torque by T. Torque is expressed in foot-pounds and inch-pounds. Some writers, in order to distinguish torque and bending moment from work, use *pound-feet* and *pound-inches* for the first two and reserve *foot-pounds*, and *inch-pounds* to mean work or energy. This distinction is not in general use.

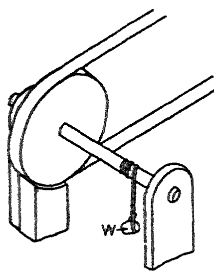


FIG. 52.—Shaft in torsion.

Problems

1. An axle 12 inches in diameter is used to lift a load of 600 pounds by means of a 1-inch rope. What is the torque? *Ans.* 3,900 inch-pounds.

2. A shaft carries a 4-foot pulley. A belt running on this pulley exerts a pull of 1,500 pounds at one point of tangency and a pull of 200 pounds at the other. Neglecting the thickness of the belt, find the torque. *Ans.* 2,600 foot-pounds.

42. Deformation and Stress at Surface of Shaft.—Fig. 53 represents a shaft fixed at the lower end. DB and EF are lines in its surface parallel to the axis CO. If the cylindrical surface between the lines DB and EF is developed, it forms the plane rectangle $DBFE$. If a torque is applied to the shaft, twisting

it in a counterclockwise direction, the point B is displaced to B' and the point F is displaced to F'. The developed surface $DBFE$ suffers a shearing deformation and becomes the parallelogram $DB'F'E$. Every point on the surface at the upper end is displaced the distance BB'. If a is the radius of the cylinder and θ (in radians) is the angle through which the top is turned with reference to the base, the displacement BB' is equal to $a\theta$.

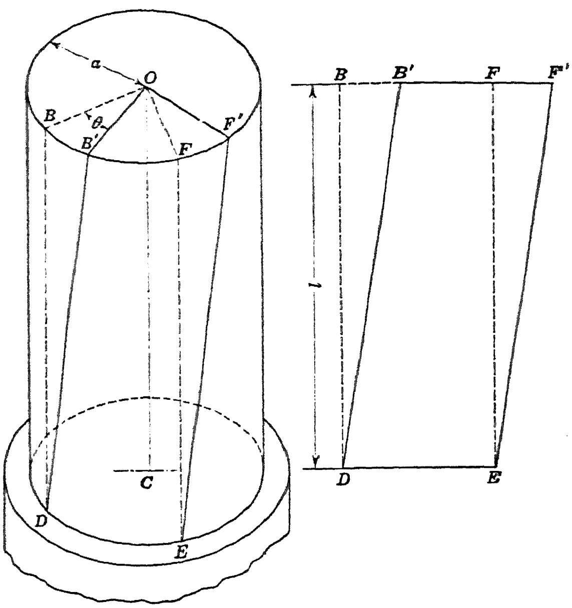


FIG. 53.—Portion of shaft in torsion.

If l is the length of the shaft, the unit shearing deformation is given by

$$\delta_s = \frac{a\theta}{l}, \tag{1}$$

and the unit shearing stress in the outer fibers is given by

$$S_s = \frac{E_s a\theta}{l}. \tag{2}$$

Problems

1. A 4-inch solid steel shaft is twisted 2 degrees in a length of 10 feet. What is the unit shearing displacement at the surface?

Ans. $\delta_s = \dfrac{\pi}{5{,}400} = 0.000582.$

2. If the modulus of rigidity of the steel of Problem 1 is 12,000,000 pounds per square inch, find the unit shearing stress in the outer fibers.

Ans. S_s = 6,984 pounds per square inch.

3. A ¼-inch rod is twisted 32 degrees in a length of 10 feet. Find the unit shearing displacement in the outer fiber.

4. If a 1-inch rod of a given material may be twisted 5 degrees in a length of 12 feet, without exceeding the allowable unit shearing stress, how much may a ⅜-inch rod of the same material be twisted in a length of 4 feet?

43. Relation of Torque to Angle of Twist.—Fig. 54 shows the upper end of the shaft of Fig. 53. An element of area dA is at the position M at a distance r from the axis. When the shaft is twisted and the top is turned through an angle of θ *radians*, this area is moved to M'. Its displacement is $r\theta$ and the unit shearing displacement is given by

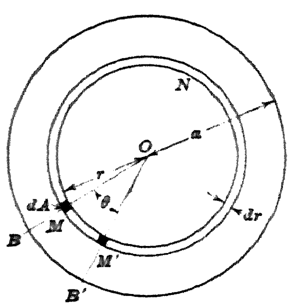


FIG. 54.—Shear displacement of torsion.

$$\delta_s = \frac{r\theta}{l}. \tag{1}$$

The unit shearing stress on dA is given by

$$s_s = \frac{E_s r\theta}{l}. \tag{2}$$

The part of the shaft between the radius r and the radius $r + dr$ is a hollow cylinder of thickness dr, which may be developed into a rectangular solid of width $2\pi r$ and thickness dr. The area of cross-section of this hollow cylinder is $2\pi r dr$. The shearing force required to deform this cylinder is the product of its cross-section multiplied by the unit shearing stress.

$$\text{Shearing force} = 2\pi r dr \times \frac{E_s r\theta}{l} = \frac{2\pi E_s\theta}{l} r^2 dr. \tag{3}$$

The moment of this shearing force with respect to the axis of the cylinder is the product of the force by the distance r.

$$\text{Resisting moment} = \frac{2\pi E_s\theta}{l} r^3 dr. \tag{4}$$

The entire shaft may be regarded as made up of a series of concentric hollow cylinders of thickness dr, and the total resisting moment, which is equivalent to the external torque, is the integral of equation (4) between the limits $r = 0$ and $r = a$.

$$T = \frac{2\,\pi E_s\theta}{l}\int r^3dr = \frac{\pi E_s\theta}{2\,l}\,[r^4]_0^a;$$

$$T = \frac{E_s\theta}{l}\,\frac{\pi a^4}{2}. \qquad (5)$$

The expression $\frac{\pi a^4}{2}$ is the polar moment of inertia of the circle of radius a and is usually represented by J. Equation (5) becomes

$$T = \frac{E_s\theta J}{l}, \qquad (6)$$

from which $\theta = \dfrac{Tl}{E_sJ}$. Formula VI.

This theory applies rigidly to circular shafts only. In Fig. 54 the straight line OMB remains straight when the shaft is twisted. In sections which are not circular, this is not the case, and the unit shearing stress is not directly proportional to the distance from the axis.

Problems

1. A 2-inch solid steel shaft is twisted 2 degrees in a length of 10 feet. If the modulus of rigidity is 12,000,000 pounds per square inch, what is the torque? *Ans.* $555.6\,\pi^2 = 5{,}483$ inch-pounds.

2. Find the angle of twist in a length of 15 feet of 4-inch solid steel shafting which drives a 6-foot pulley when the tension on the belt at one point of tangency is 2,900 pounds and at the other point 300 pounds. The modulus of rigidity is 11,700,000. *Ans.* $\theta = 3.28$ degrees.

3. A hollow tube 2 inches outside diameter and 1 inch inside diameter is twisted by a force of 120 pounds at the end of an arm 5 feet long, and the angle of twist is 1 degree in a length of 40 inches. Find E_s.
Ans. $E_s = 11{,}200{,}000$ pounds per square inch.

44. The Relation of Torque to Shearing Stress.—From equation (6) of Article 43,

$$T = \frac{E_s\theta J}{l}; \qquad (1)$$

and from equation (2) of Article, 43

$$S_s = \frac{E_sa\theta}{l}; \qquad (2)$$

from which $\dfrac{E_s\theta}{l} = \dfrac{S_s}{a}$. (3)

Substituting in (1)

$$T = \frac{S_sJ}{a}, \qquad (4)$$

$$S_s = \frac{Ta}{J} \qquad \text{Formula VII.}$$

For a solid circular shaft, $J = \frac{\pi a^4}{2}$, and $\frac{J}{a} = \frac{\pi a^3}{2}$;

$$S_s = \frac{T}{\frac{\pi a^3}{2}} = \frac{16\,T}{\pi d^3}. \tag{5}$$

Formula VII gives the unit shearing stress at the surface of a ~~solid~~ *circular shaft* of radius a subjected to a torque of T inch-pounds.

For a hollow circular shaft of inside radius b,

$$J = \pi\frac{(a^4 - b^4)}{2} = \frac{A\,(a^2 + b^2)}{2};$$

$$S_s = \frac{2\,Ta}{\pi(a^4 - b^4)}.$$

Problems

1. A 2-inch solid shaft is twisted by a force of 300 pounds applied to a wrench at a distance of 4 feet from the axis of the shaft. Find the unit shearing stress at the outer surface of the shaft.

Ans. $S_s = \frac{14{,}400 \times 2}{\pi} = 9{,}167$ pounds per square inch.

2. A 4-foot flywheel is driven by a 3-inch solid shaft. The tension on part of the belt is 1,500 pounds, and on the other 300 pounds. Find the torque and the maximum unit shearing stress.

Ans. $S_s = 5{,}432$ pounds per square inch.

3. A 4-inch solid shaft exerts a torque of 6,000 foot-pounds. Find the maximum unit shearing stress. *Ans.* 5,729 pounds per square inch.

4. Solve Problem 3 if the shaft is hollow with the outside diameter 4 inches and the inside diameter 2 inches.

Ans. $S_s = \frac{72{,}000 \times 2 \times 2}{\pi\,(16 - 1)} = 6{,}111$ pounds per square inch.

5. Solve Problem 3 if the shaft is hollow with the inside diameter 2 inches and the outside diameter such that the area of cross-section is equivalent to that of a 4-inch solid shaft. *Ans.* 4,271 pounds per square inch.

Formula VII may be derived in another way. From Fig. 54 it is seen that the displacement of any area dA is proportional to its distance r from the axis. Hence the unit shearing stress is also proportional to the distance of the element from the axis of the shaft. If s_1 is the unit shearing stress at unit distance from the axis, the unit shearing stress at a distance r is s_1r.

The shearing stress on an area dA is s_1rdA and the resisting moment is s_1r^2dA. The total moment is given by

$$T = s_1\int r^2dA; \tag{1}$$

but $\int r^2dA$ is the polar moment of inertia J, from which

$$T = s_1J. \tag{2}$$

But since s_s at any point is equal to s_1r, the unit shearing stress at the surface where $r = a$ is given by $S_s = s_1a$, hence $s_1 = \frac{S_s}{a}$, which substituted in (2) gives $T = \frac{S_sJ}{a}$. Formula VII.

45. Relation of Torque to Work.—To an arm of length R, measured from the axis of a shaft, a force P is applied which is perpendicular to the plane passing through the axis of the shaft and the point of application of the force. The torque is RP. When the shaft makes one revolution, the point of application of the force moves through a distance $2\pi R$. The work done by the force P in making one revolution is $2\pi RP$. Since PR is the torque

$$\underset{\text{Per Rev}}{\text{Work}} = 2\pi RP = 2\pi T.$$

The work per revolution is 2π times the torque. This is true whether the torque is due to a single force or to a number of forces.

In problems relating to work done by a rotating body, solve first for the torque. When this is obtained in a numerical or literal equation, it may be used in Formulas VI and VII.

Problems

1. A shaft transmits 600 horsepower at 180 revolutions per minute. What is the torque in foot-pounds? *Ans.* 17,506 foot-pounds.

2. In Problem 1, what must be the diameter of the shaft if the maximum unit shearing stress is 5,000 pounds per square inch?

Ans. 6 inches nearly.

3. How many horsepower may be transmitted by a hollow shaft 8 inches inside diameter and 12 inches outside diameter, if the maximum shearing stress is 6,000 pounds per square inch and the speed is 100 revolutions per minute?

4. If S_s is the allowable unit shearing stress, N is the number of revolutions per minute, hp is the horsepower, and a is the radius of the shaft, show that

$$a^3 = \frac{33{,}000 \times 12 \times hp.}{\pi^2NS_s} = \frac{40{,}123\ hp}{NS_s}.$$

5. If the allowable unit shearing stress is 5,000 pounds per square inch, show that the diameter of a solid shaft should be approximately,

$$d = 4\sqrt[3]{\frac{hp.}{N}}.$$

46. Helical Springs.—An interesting example of torsion is the stretching or compression of a helical spring, such as is shown in Fig. 55. A helical spring is made by winding a wire or rod on a cylinder (in a single layer, usually). The radius of the coil of the spring is the sum of the radii of wire and the cylinder about which it is wound. The ends of the wire, when the spring is to be used in tension, are turned in to the center so as to apply the force in the line of the axis.

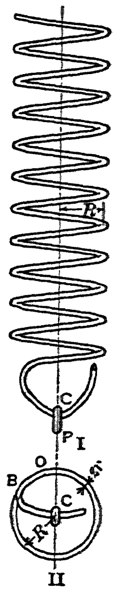


FIG. 55.—Helical spring.

The greater part of the elongation of a helical spring is due to torsion. If we consider a section at O, Fig. 55, II, we find that there is a twisting moment PR due to the load P at the axis. (Fig. 55, II, is a plan of the lower turn; the force P is normal to the plane of the paper.) The point C at which the load P is applied being at the center of the circle, it lies in the plane of the section and therefore there is no bending moment. The effect of a force acting on an arm CBO is independent of the form of the arm. As far as concerns the stresses at the section, CBO might be a straight rod from C to O. The point O is any point in the spring except the portion CB and the similar portion at the top. With these exceptions the entire spring is subjected to a twisting moment PR. In addition to this torsion, there is a constant total shear P. Also, since the turns of the coils are not exactly horizontal, there is another slight correction. Both of these are neglected in ordinary calculations.

To get the elongation of a helical spring due to a given load, multiply the angle of twist in the entire spring by the radius R.

in radians always.

Problems

1. A rod 0.2 inch in diameter is used to make a helical spring of 20 turns. The radius of the coil from the axis to the center of all sections is 1 inch. What is the elongation, due to a load of 3 pounds, if the modulus of rigidity is 12,000,000 pounds per square inch.

P. T = 3 inch-pounds; $J = \frac{0.0001\pi}{2}$; length of rod, 40π.

'2'

Torque = $P \times R$ (R is lever-arm)

gets J.

$def = R\theta$
$= \frac{RTl}{EJ}$
$= R \cdot PR \cdot 2\pi Rn \times 2$

$$\theta = \frac{3 \times 40\pi \times 2}{12{,}000{,}000 \times 0.0001\,\pi} = 0.2 \text{ radian.}$$

$$\text{Elongation} = 0.2 \times 1 = 0.2 \text{ inch.}$$

2. What is the unit shearing stress in Problem 1?

Ans. $S_s = \frac{6{,}000}{\pi} = 1{,}910$ pounds per square inch.

3. If the same rod were used to make a spring of ten turns, each of 2-inch radius, what would be the elongation due to a load of 3 pounds, and what would be the unit shearing stress?

Ans. 0.8 inch, 3,819 pounds per square inch.

4. At Watertown Arsenal, a steel rod 1.24 inches in diameter and about 241 inches long was formed into a helical spring 7.64 inches *outside* diameter. A load of 5,000 pounds shortened this spring 4.64 inches. Find the modulus of shearing elasticity. *Ans.* 11,460,000.

5. In Problem 4 find the unit shearing stress under the load of 5,000 pounds. *Ans.* 42,740 pounds per square inch.

6. If R is the radius of the helix, r the radius of the rod, P the load, E_s the modulus of elasticity in shear, and n the number of turns, prove that

$$\text{Elongation} = \frac{4PR^3n}{E_s r^4}.$$

7. If S_s is the allowable unit shearing stress, find the elongation of a spring in terms of S_s, E_s, R, r, and n. *Ans.* $\text{Elongation} = \frac{2\pi S_s R^2 n}{E_s r}.$

8. Find the expression for the elongation of a helical spring in terms of S_s, E_s, R, r, and l, where l is the length of the rod. *Ans.* $\text{Elongation} = \frac{S_s R l}{E_s r}.$

47. Resilience in Torsion.—A force P at the end of an arm R twists a shaft of length l through an angle of θ radians. If there is no torque at the beginning, the average force is $\frac{P}{2}$, and the work of twisting is given by

$$\text{Work} = \frac{PR\theta}{2} = \frac{T\theta}{2} = U, \tag{1}$$

$$U = \frac{T^2 l}{2E_s J}. \tag{2}$$

If the shaft is circular and of radius a,

$$U = \frac{J^2 S_s^2 l}{2a^2 E_s J} = \frac{J S_s^2 l}{2a^2 E_s}. \tag{3}$$

For a solid circular shaft

$$U = \frac{\pi a^2 l S_s^2}{4\,E_s} = \frac{S_s^2}{4\,E_s} \times \text{volume}, \tag{4}$$

and the energy per unit volume $= U = \frac{S_s^2}{4\,E_s}$. Formula VIII.

Problems

1. In Problem 4 of Article 46 find the work done by the load of 5,000 pounds in shortening the spring and the work per cubic inch.

Ans. 11,600 inch-pounds.
39.8 inch-pounds per cubic inch.

2. Find the resilience per cubic inch in Problem 1 by means of equation [illegible] and the answers of Problems 5 and 4 of Article 46.

3. A spring at the Watertown Arsenal was made of 36 pounds of steel rod 1.02 inches in diameter. The outside diameter of the coil was 4.30 inches. A load of 11,000 pounds changed the length of this spring from 20.63 inches to 16.67 inches. After the load was removed the spring returned to its original length to within 0.02 inch. Find the energy per cubic inch and the energy per pound. *Ans.* 50.4 foot-pounds per pound.

4. In Problem 3 what was the maximum shearing stress due to torsion?

Ans. 86,580 pounds per square inch.

CHAPTER VI

BEAMS

48. Definition of a Beam.—Fig. 56 is a front elevation of a beam supported near the ends and carrying a single *concentrated* load P in addition to its own weight. If the beam is uniform, its own weight is a *uniformly distributed* load. There is an upward reaction at each support. A beam may be defined as a rigid body subjected to transverse loads and reactions.

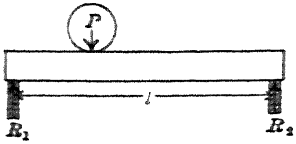


FIG. 56.—Beam supported at ends.

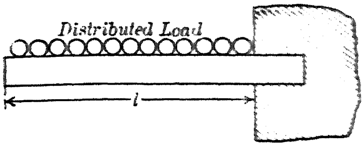


FIG. 57.—Cantilever.

49. Kinds of Beams.—Beams may be classified according to the character of the support and the method of loading. Fig. 57 represents a beam fixed at one end and free at the other. This kind of beam is called a *cantilever*. Fig. 58 is a beam *fixed* at both ends. Fig. 59 is *fixed* at the right end and *supported* at the left. Fig. 60 is a beam which *overhangs* its supports.

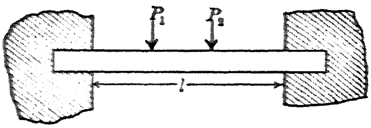


FIG. 58.—Beam fixed at both ends.

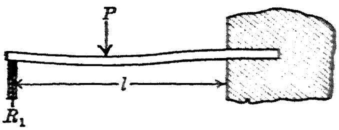


FIG. 59.—Beam fixed at one end and supported at the other.

A beam with three or more supports, as in Fig. 61 is a *continuous*, beam.

The figures show different methods of loading and some of the ways of representing the loads and reactions in diagrams and drawings.

In Figs. 56 and 59, we have a single concentrated load. In

Fig. 58, there are two concentrated loads. In Fig. 57, there is a uniformly distributed load over the entire length. In Fig. 60, there is a uniformly distributed load over half of the length, and a concentrated load at the right end. In Fig. 61, there is a distributed load over part of the left portion and another load of greater weight per unit length over the right half.

A beam is not necessarily horizontal. A vertical fence post subjected to the horizontal force of the wind or the weight of

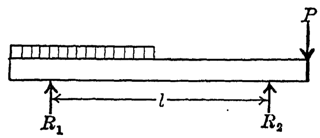


FIG. 60.—Beam overhanging its supports.

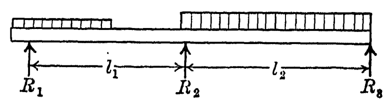


FIG. 61.—Continuous beam.

a gate is a cantilever beam. A post at the end of a line of wire fence is a vertical beam supported at one end and partially fixed at the other, with several concentrated loads due to the tension in the wire.

50. Reactions at Supports.—The calculation of the reactions at the supports of a beam is a problem of mechanics, involving the equilibrium of non-concurrent, coplanar forces. It will be remembered that in the general case three independent equations

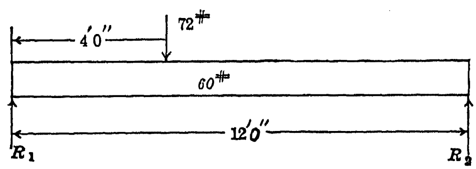


FIG. 62.—Beam supported at ends.

may be written and that the problem can be solved when there are three unknown forces. When the forces are all parallel, as in most cases of beams, there may be only two independent equations and two unknown forces. One of these equations must be a moment equation and the other may be either a moment or a resolution equation. The moment equations should be taken about axes which intersect unknown forces. In this way one unknown force will have no moment arm and will be eliminated from the equation.

Example

A uniform beam 12 feet long, weighing 60 pounds, is supported at the ends and carries a load of 72 pounds 4 feet from the left support (Fig. 62). Find the reactions at each support. Remembering that the center of gravity of a uniform beam is at the middle of its length, take moments about a horizontal line perpendicular to the beam through the right support.

Load in pounds		Arm in feet		Moment in foot-pounds
60	×	6	=	360
72	×	8	=	576
R_1	×	12	=	936

$R_1 = 78$ pounds.

Taking moments about the left support:

$$60 \times 6 = 360$$
$$72 \times 4 = 288$$
$$R_2 \times 12 = 648$$
$$R_2 = 54 \text{ pounds.}$$

Check by vertical resolutions.

Loads	Reactions
60	54
72	78
132	132

Example

A beam 20 feet long, weighing 30 pounds per foot is supported at the right end and 4 feet from the left end and carries a load of 80 pounds at the left end (Fig. 63). Find the reactions, and check.

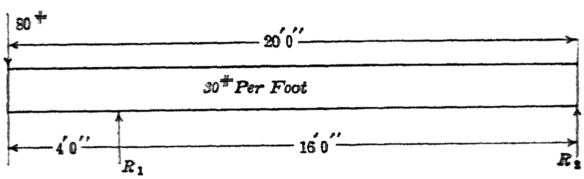


FIG. 63.—Beam overhanging left support.

The total weight of the beam is $30 \times 20 = 600$ pounds.
Taking moments about an axis through the right support.

$$600 \times 10 = 6{,}000$$
$$80 \times 20 = 1{,}600$$
$$R_1 \times 16 = 7{,}600$$
$$R_1 = 475 \text{ pounds.}$$

Taking moments about an axis through the left support,

$$\begin{aligned} 600 \times \quad 6 &= 3{,}600 \\ 80 \times -4 &= -320 \\ \hline R_2 \times \quad 16 &= 3{,}280 \\ R_2 &= 205 \text{ pounds.} \end{aligned}$$

It will be noticed that in the first part of each example counterclockwise moment is written positively and in the second part clockwise is written positively. This is done for convenience to avoid negative signs as much as possible. It makes no difference which sign is given to a moment expression, provided the same convention is retained throughout one equation. When the moments are not all of the same sign, it is convenient to take as positive the rotation which has the greatest number of terms. The direction of a moment should always be determined by noting which way it tends to rotate about the axis of moments rather than by observing the mathematical sign of the forces and the arms.

In the second example, the moment of the left 4 feet of the beam is counterclockwise about the left support while that of the remaining 6 feet is clockwise. Some students would write these two portions separately taking 120 pounds with a moment arm of 2 feet and 480 pounds with a moment arm of 8 feet. The method used in the illustrative example, where the whole weight is treated as concentrated at its center of gravity 6 feet from its right support, is shorter. Again, some would write these moments in the form of an equation, the first part of the second example being

$$600 \times 10 + 80 \times 20 = 16\,R_1.$$

This is sometimes convenient when there are factors which can be cancelled, but generally it is better to arrange the work as shown in the example. Where there are a large number of terms, several of which are negative, it is advisable to put the positive moments in one column and the negative moments in another.

Problems

1. A uniform beam, 24 feet long, weighing 60 pounds per foot, is supported 2 feet from the left end and 6 feet from the right end, and carries a load of 240 pounds on the left end and a load of 320 pounds 8 feet from the left

end. Find the reactions by two moment equations and check by vertical resolutions.

2. A horizontal beam 20 feet long, weighing 80 pounds, is supported 4 feet from the right end and held by a downward reaction at the right end. Find these reactions and check.

3. A beam 4 feet long, weighing 60 pounds, with its center of gravity at the middle, is hinged at the lower right corner (Fig. 64) and held horizontal by a horizontal pull 8 inches above the hinge. Find this horizontal pull (H), the horizontal component of the hinge reaction (C), and the vertical component of the hinge reaction (V).

Ans. H, 180 pounds; C, 180 pounds; V, 60 pounds.

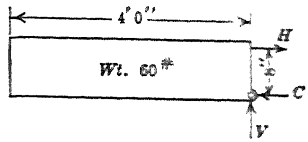


FIG. 64.—Beam supported by horizontal couple.

51. Shear in a Beam.—Figs. 65 and 69 show a horizontal cantilever weighing w pounds per foot with a vertical load of P pounds at a distance of a feet from the free end. A section EFG perpendicular to the length of the beam is taken at a distance of x feet from the free end. The portion of the beam to the left of this section will be treated as a free body in equilibrium. The weight of this portion is wx pounds, and together with the load of P pounds constitute the *external load* on the portion of beam. It is kept in equilibrium by the internal *forces* which the right portion exerts on the left portion across the plane EFG.

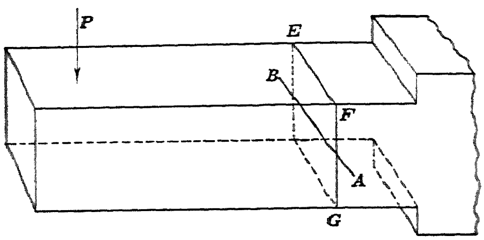


FIG. 65.—Section of cantilever.

Fig. 66 shows the beam actually cut in two at the section FG. A cylinder, with its axis horizontal and perpendicular to the length of the beam, is put between the two portions near the bottom, and a short horizontal chain connects them near the top. A second chain attached to the right end of the left portion runs vertical.

Since the forces exerted by the cylinder and the horizontal chain are horizontal they have no vertical component, so that

the tension in the vertical chain must be equal to the resultant of the external loads. This is easily shown to be true if the vertical chain is supported by a spring balance. By horizontal resolutions it is seen that the pull H in the horizontal chain is equal to the compression C in the cylinder.

In Fig. 67 the cylinder is replaced by a rectangular block. If the coefficient of friction is sufficiently large, the friction will exert a vertical force equal to the weight of the portion, and the

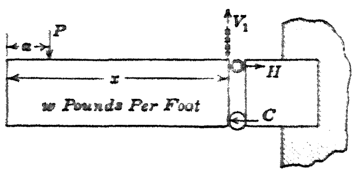


FIG. 66.—Cantilever shear and tension.

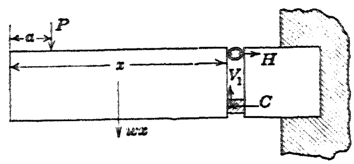


FIG. 67.—Cantilever shear resisted by friction.

vertical chain may be removed. This vertical force is transmitted across the rectangular block as vertical shear.

Fig. 68 shows the same beam with the pieces glued together. The lower part of the glue is in compression, the upper part is in tension, and all of it is in shear.

Fig. 69 represents a similar beam which has not been cut. The material in it at any section is under the same stresses as the glue of Fig. 68.

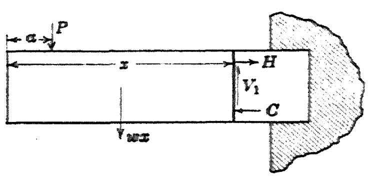


FIG. 68.—Resisting shear and moment at glued section.

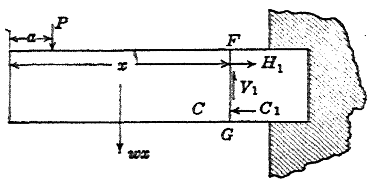


FIG. 69.—Shear and moment at section.

The vertical shear, V_1 of Fig. 69, is called the *resisting shear.* The resultant of all the forces parallel to the section which act on the portion of the beam on either side of the section is called the *external shear.* In a horizontal beam the external shear (for a section at right angles to the beam) is vertical and is called the *total vertical shear.* In formulas total vertical shear is represented by V. The resisting shear on one side of any section is equal and opposite to the external shear acting on the portion of the beam

on the other side of the section. In Fig. 68 the external shear on the portion of the beam to the left of the section is $P + wx$ acting downward and is equal to the resisting shear with which the portion to the right of the section acts on the glue. Since the entire beam is in equilibrium under the action of the external forces, the external shear on the portion to the right of the section must be equal and opposite to the shear on the left portion. In like manner, the portion to the left of the section exerts a shear equal and opposite to V_1 upon the portion to the right.

The *magnitude* of the vertical shear may be determined from the vertical resolution of all the external forces which act on either the left or the right portion of the beam. In Figs. 68 and 69 it is convenient to use the left portion, since the *external* forces on this portion are given. In a cantilever *fixed* at the *left end* it would be better to consider the portion to the right of the section as the free body in equilibrium.

The *sign* of the vertical shear is determined from the resultant of the forces which act on the portion to the *left* of the section. The vertical shear is positive when the resultant of the vertical forces to the left of the section is upward. It is positive when the *internal* shear acting from the left portion upon the right portion at the section is upward. In Figs. 68 and 69, the vertical shear is negative.

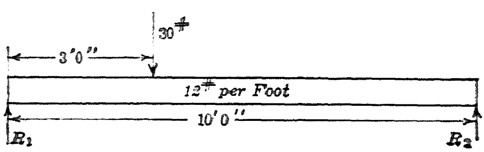


FIG. 70.—Beam supported at ends.

Example

A uniform horizontal beam, 10 feet long, weighing 12 pounds per foot, is supported at the ends and carries a load of 30 pounds 3 feet from the left end (Fig. 70). Find the total vertical shear at a section 2 feet from the left end and at a section 4 feet from the left end.

The reactions are 81 pounds at the left support and 69 pounds at the right support. Using the left portion as the free body in equilibrium, at 2 feet from the left end,

$$V_2 = 81 - 2 \times 12 = 57 \text{ pounds.}$$

At 4 feet from the left end,

$$V_4 = 81 - 4 \times 12 - 30 = 3 \text{ pounds.}$$

Using the right portion as the free body,

$$-V_4 = 69 - 6 \times 12 = -3 \text{ pounds.}$$

When the right portion was used as the free body the result is negative, but since it is agreed that the left portion shall determine the sign, this must be called $-V_4$, and

$$V_4 = 3 \text{ pounds}$$

as before.

Problems

1. In the above example, find the vertical shear at 1 foot from the left end and at 9 feet from the left end.

Ans. $V_1 = 69$ pounds, $V_9 = -57$ pounds.

2. A horizontal cantilever, fixed at the right end, weighs 60 pounds per foot. A rope attached at the left end exerts a pull of 100 pounds upward at an angle of 30 degrees to the vertical. Find the total vertical shear at 1 foot and at 2 feet from the left end.

Ans. $V = 26.6$ pounds at 1 foot, $V = -33.4$ pounds at 2 feet.

52. Bending Moment and Resisting Moment.—To calculate the compressive and tensile forces at any section one moment equation must be written. The moments are calculated with respect to an axis in the plane of the section FG (Figs. 71 and 69) perpendicular to the external forces. (In Fig. 69 this axis

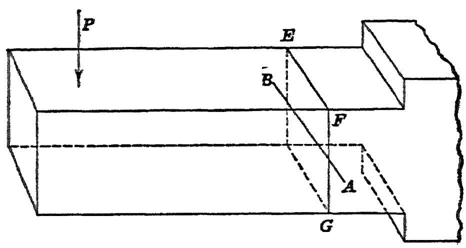


FIG. 71.—Moment of cantilever section.

is perpendicular to the plane of the paper.) Considering the portion of the beam to the left of the section (Fig. 69) as a free body, the *moment* of the *resisting shear* is zero (since its line of action passes through the axis of moments) and the moment of the *external forces* to the left of the section must be equal and opposite to the moment of the forces, H_1 and C_1, which act across the section from the right portion. It is advisable to take the axis of moments through the center of gravity of the cross-section. The horizontal line AB through the center of gravity of the section EFG of Fig. 71 is such an axis of moments. The moment of the external forces on the free portion of the beam about this axis is the *external moment* at the section, or the *bending moment*. The moment of the internal forces parallel to the

length of the beam about the same axis is called the *resisting moment* or internal moment. As these two moments are always equal, it is customary to speak of the *moment* at a *section* without distinguishing between internal and external moment. When the external reactions and loads are all perpendicular to the length of the beam, and, consequently, parallel to the section, the moment arms of all these forces may be measured from the plane of the section, and it is not necessary to consider the axis of moments as passing through the center of gravity. As this is the usual case we are accustomed to speak of the "moment at a section," without considering any particular axis in that section.

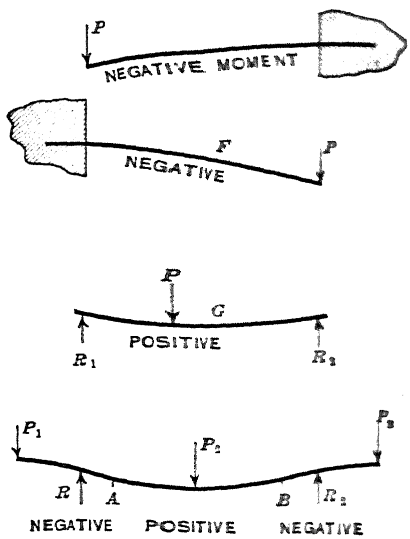


FIG. 72.—Positive and negative moment.

The bending moment is considered positive at any section when the *portion* to the *left* of the *section* tends to turn the *portion* to the *right* in a *clockwise* direction, or when the *resisting moment* from *right* to *left* is *counterclockwise*. The bending moment is negative in Figs. 69 and 71 and is positive in Fig. 70. In the cantilever fixed at the left end in Fig. 72 the moment at all sections is negative. If we consider any section F of the cantilever fixed at the left end we see that the portion to the right of F tends to turn the portion to the left of F in a clockwise direction, consequently the portion to the right is turned counterclockwise by the portion to the left and the moment is also negative. In the beam supported at the ends the part between the left support and the load is evidently tending to turn the remainder of the beam in a clockwise direction. Between the load and the right support the moment of the left reaction is positive while the moment of the load is negative, but with the beam supported as shown in Fig. 70 the moment of the reaction is the greater, so that the resultant moment is positive. This is seen to be the

case if the portion to the right of the section G is considered as the free body. The reaction at the right support turns this portion counterclockwise, consequently the moment of the left portion is clockwise and positive.

The beam which overhangs the supports has negative moment from the left end to a point A, which is some distance to the right of the left support, and from the point B to the right end. Between A and B the moment is positive.

From the bent beams of Fig. 72 it is seen that with negative moment the beam is convex upward and that the center of curvature is downward. With the positive moment the beam is concave upward. The most convenient method of determining the sign of the moment in many cases is by means of the center of curvature. If the center of curvature is on the positive side (above the beam) the moment is positive; if the center of curvature is on the negative side of the beam the moment is negative.

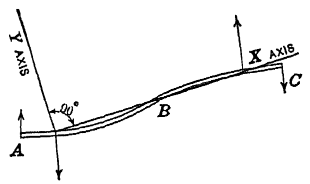


FIG. 73.—Inclined beam.

When a beam is not horizontal, as in Fig. 73, the X axis is taken parallel to the direction of its length and the Y axis at 90 degrees therefrom in a counterclockwise direction. The moment is positive when the ordinate of the center of curvature is positive and *vice versa*. In Fig. 73 the moment is positive from A to B and negative from B to C.

In formulas and equations, the bending moment is represented by M.

Example

A uniform horizontal beam 10 feet long, weighing 24 pounds per foot, is supported at the ends and carries a load of 60 pounds 3 feet from the left end. Find the moment and shear at 2 feet from the left end and at 5 feet from the left end.

The left reaction is 162 pounds and the right reaction is 138 pounds. At 2 feet from the left end the moment is that of the left reaction with an arm of 2 feet turning clockwise, and the weight of 2 feet of beam with a moment arm of 1 foot turning counterclockwise.

Force		Arm		Moment
162	$\times$	2	=	324
−48	$\times$	1	=	−48
$V =$ 114 pounds				$M =$ 276 foot-pounds.

At 5 feet from the left end,

Force in pounds		Arm in feet	Moments in foot-pounds	
162		5.0	810	
	−120	2.5		−300
	− 60	2.0		−120
	−180			−420

$V_5 = -18$ pounds. $M_5 = 390$ foot-pounds.

Problems

1. A horizontal cantilever, weighing 40 pounds per foot, is fixed at the right end and carries a load of 60 pounds 3 feet from the left end. Find the vertical shear and the moment at 2 feet from the left end and at 4 feet from the left end. *Ans.* V_2, −80 pounds; M_2, −80 foot-pounds; V_4, −220 pounds; M_4, −380 foot-pounds.

2. A uniform horizontal beam, 15 feet long, weighing 24 pounds per foot, is supported at the right end and 3 feet from the left end. Find the total vertical shear and the moment at 2 feet from the left end and at 6 feet from the left end. Check the result at 6 feet from the left end by using the part to the right of the section as a free body.

3. A beam of length l, supported at the ends, carries a load P at the middle. Find the moment at the middle. *Ans.* Moment at the middle, $\frac{Pl}{4}$.

4. A beam of length l, supported at the ends, carries a uniformly distributed load of w pounds per unit length. Find the moment at the middle and the shear at the middle. Compare results with handbook.

5. Find the moment at the fixed end of a cantilever of length l due to a load P at the free end. Also find the moment due to a uniform load W. Find the shear in each case at the fixed end.

Ans. Moments, $-Pl$, $-\frac{Wl}{2}$; shear, $-P$, $-W$.

6. If an observer should pass from one side of a beam to the other (from front to rear), show that the sign of the shear as viewed from the new position will be reversed but the sign of the moment will not be changed.

53. Shear Diagrams.—It is convenient to represent the total shear at all sections of a beam by means of a shear diagram. Fig. 74 is the shear diagram of a uniform horizontal beam 12 feet long weighing 20 pounds per foot, which is supported at the ends. The end reactions are each 120 pounds. Begin with a section infinitely close to the left support. The weight of the portion of beam to the left of this section is negligible so that the shear is the left reaction of 120 pounds. Infinitely close to the right support the shear is the left reaction of 120 pounds less the weight of practically all the beam, which is 240 pounds, so that

its value at this section is -120 pounds. At 1 foot from the left support the shear is 100 pounds; at 2 feet it is 80 pounds. If we construct the shear diagram we observe that the curve which connects these points is a straight line so that it is only necessary

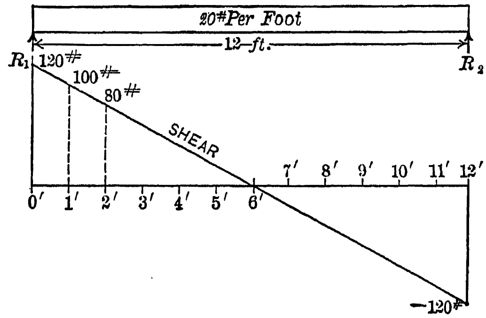


FIG. 74.—Shear diagram for distributed load.

to calculate the shear at the ends and connect the corresponding points on the diagram by a straight line. This straight line and the ordinates at the ends make the shear diagram.

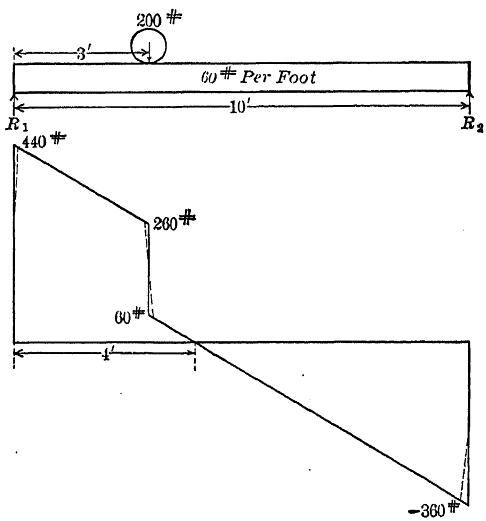


FIG. 75.—Concentrated and distributed loads.

Fig. 75 is the shear diagram for a beam 10 feet long, supported at the ends with a uniform load due to its own weight of 60 pounds per foot and a concentrated load of 200 pounds, 3 feet from the left end. By moments about the right support, we

find the left reaction, R_1, to be 440 pounds. By moments about the left support, we find R_2 to be 360 pounds. The sum of these reactions is equal to the total load, affording a check.

Infinitely close to the left support the shear is 440 pounds. It drops 180 pounds in the first 3 feet and is 260 pounds infinitely close to the load of 200 pounds. In a negligible distance in passing from the left side to the right side of the concentrated load it drops an additional 200 pounds, so that it becomes 60 pounds (260 - 200) infinitely close to the load on the right side. Here the shear diagram is a vertical line. Beyond the concentrated load the shear drops at the rate of 60 pounds per foot for the remaining 7 feet, which brings it to -360 pounds infinitely close to the right support. The reaction of 360 pounds raises it to the initial line. The shear diagram crosses the zero or initial line 1 foot to the right of the concentrated load, or 4 feet from the left support.

Notice that the shear diagram in Fig. 75 drops vertically downward under the load of 200 pounds, and we speak of points as infinitely near the load on either side. This would mean that the load is applied along a mathematical line running across the beam. The actual surface of contact is a band of some width running across the beam, and the actual shear diagram is something like that represented by the dotted lines.

It is often desirable to write the equation of the shear diagram in terms of the loads and reactions and the distance along the beam from some fixed point. In Fig. 74 the shear decreases 20 pounds per foot and is 120 pounds at the left end. If x is the distance of any section from the left end taken as the origin the equation is

$$V = 120 - 20\,x.$$

In Fig. 75 the shear decreases 60 pounds per foot. The equation for the first 3 feet is

$$V = 440 - 60\,x, \tag{1}$$

and for the remainder of the beam

$$V = 440 - 200 - 60\,x = 240 - 60\,x. \tag{2}$$

Problems

1. A uniform horizontal beam 12 feet long, weighing 10 pounds per foot, is supported at the left end and 2 feet from the right end and carries a load of 15 pounds 2 feet from the left end. Construct the shear diagram, using as abscissas 1 inch equals 2 feet of length, and as ordinates 1 inch equals 20 pounds shear. Write the equation of the three parts of this diagram with the left end as the origin.

2. Solve equations (1) and (2) above for the position of zero shear of Fig. 75. Which equation gives the correct result? What is the meaning of the result of the other equation?

3. Solve the equations of Problem 1 for the position of zero shear.

Shear diagrams are usually made up of straight lines. These lines are horizontal from one load to the other when the loads are concentrated and the weight of the beam is neglected. With uniformly distributed loads, the lines slope downward from left to right. (With distributed loads pushing up, as in the bottom of a boat due to the water pressure, the lines slope upward.) Where loads are distributed not uniformly, as in the case of the water pressure on a vertical dam, the shear diagram is curved.

The student should become sufficiently familiar with the simpler shear diagrams to be able to recognize the character of the loading at a glance.

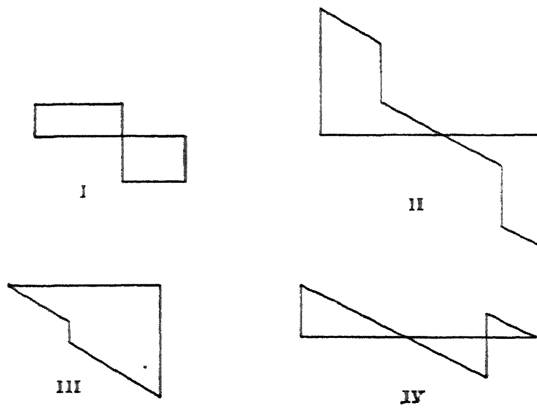


FIG. 76.—Shear diagrams.

Problem

4. Describe the loading and the character of support which gives each of the shear diagrams of Fig. 76.

54. Moment Diagrams. Moment diagrams are constructed in the same way as shear diagrams, using external moment as ordinates. In this book we shall draw positive moment upward, though many engineers prefer the opposite.

Shear diagrams are easily constructed, as they usually consist of straight lines. Moment diagrams are curved, except when the loading is made of concentrated loads only.

Fig. 77 shows the shear and moment diagram for a beam supported at the ends and carrying a load P at the middle. The weight of the beam is neglected. The end reactions are $\frac{P}{2}$. The moment at any section at a distance x from the left end is $\frac{Px}{2}$, provided x is not greater than one-half of the length. Under the load the moment is $\frac{Pl}{4}$. The moment diagram for the left half of the beam is a straight line through the points (0, 0) $\left(\frac{l}{2}, \frac{Pl}{4}\right)$. Beyond the load, the moment is due to the reaction at the left

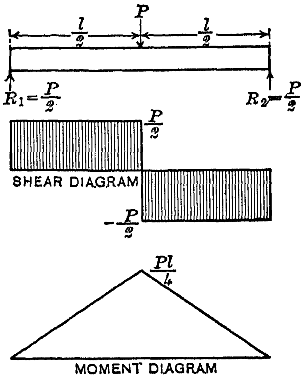


FIG. 77.—Single concentrated load.

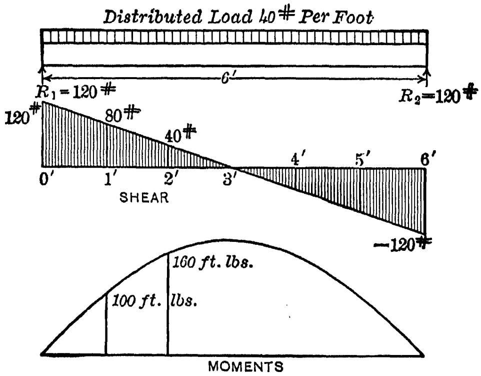


FIG. 78.—Uniformly distributed load.

support turning clockwise minus that due to the load at the middle. At a distance x from the left end, when x is greater than $\frac{l}{2}$,

$$\text{Moment} = \frac{Px}{2} - P\left(x - \frac{l}{2}\right) = \frac{Pl}{2} - \frac{Px}{2} = \frac{P}{2}(l - x).$$

This is also a straight line. Notice that the last of the three expressions for moment is the one which we get directly, if we use the portion to the right of the section as a free body. The right reaction is $\frac{P}{2}$ and the moment arm is $l - x$. The sign is opposite, as it should be.

Fig. 78 gives the shear and moment diagrams for a beam supported at the ends, with a uniformly distributed load. The moment diagram is a parabola with the vertex at the top.

Problems

1. With the data of Fig. 78, find the equation of the moment curve.

2. Find the equation of the moment curve of a beam supported at the ends, with a uniformly distributed load of w pounds per unit length.

$$Ans. \quad M = \frac{wlx}{2} - \frac{wx^2}{2} = \frac{wx}{2}(l - x).$$

3. A beam 12 feet long, supported at the ends, carries a concentrated load of 150 pounds 4 feet from the left end. Neglecting the weight of the beam, construct the shear and moment diagrams to the scale of 1 inch horizontal equals 2 feet of length, and 1 inch vertical equals 100 pounds shear and 100 foot-pounds moment. Arrange the diagrams and the sketch of the beam as in Figs. 77 and 78.

4. A beam 12 feet long, weighing 20 pounds per foot, is supported at the left end and 2 feet from the right end and carries a load of 100 pounds 2 feet from the left end and a load of 80 pounds at the right end. Construct the shear and moment diagrams, using the scale of Problem 3.

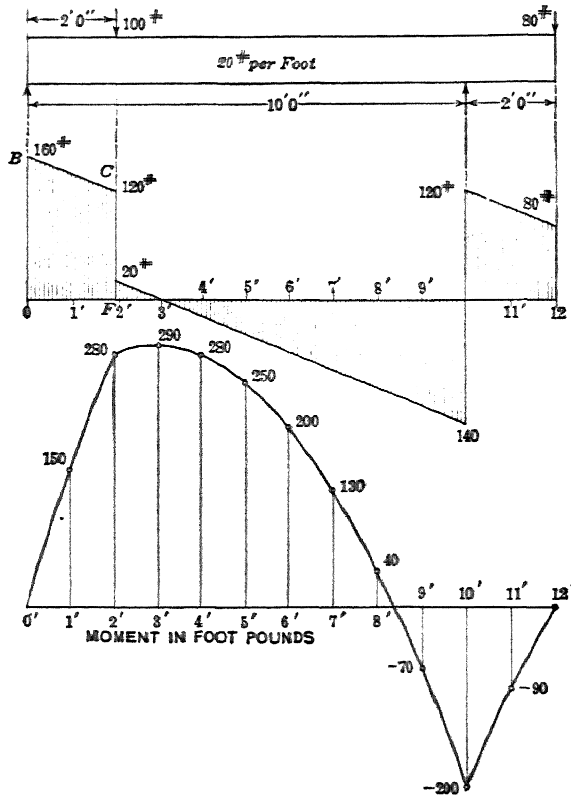


FIG. 79.—Shear and moment diagrams.

Fig. 79 shows the curves of Problem 4. Under the load of 100 pounds the shear drops from 120 pounds to 20 pounds, and there is an abrupt change in curvature in the moment diagram. Also at the right support the shear diagram rises vertically and the moment curve again changes. The shear diagram crosses the X axis at 3 feet from the left end, at which point the moment is a maximum, and again at the right support where the moment is a minimum.

5. Write the equation of the moment curve of Problem 4 in terms of the distance from the left end.

$$Ans. \begin{cases} M = 160\,x - 10\,x^2, \text{ from 0 to 2 feet.} \\ M = 60\,x + 200 - 10\,x^2, \text{ from 2 feet to 10 feet.} \\ M = 320\,x - 2{,}400 - 10\,x^2, \text{ from 10 feet to 12 feet.} \end{cases}$$

6. Find the moment at 2 feet from the left end by both the first equation and second equation of Problem 5, and find the moment at the right support by the second equation and third equation.

7. Solve the second equation of Problem 5 for the position where the moment is zero. There are two solutions of the algebraic equation. Which one should be taken? Why?

8. Apply the mathematical condition of maximum and minimum to the second equation of Problem 5 to find the position of maximum moment.

55. The General Moment Equation.—In the examples which have been given, the origin of coördinates has been taken at the left end of the beam. But it is often desirable to be able

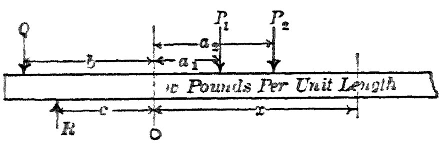


FIG. 80.—General case of loading.

to write the moment equation with any point as the origin. Fig. 80 represents a beam of indefinite extent with the origin of coördinates on a vertical line through O. To the right of the origin, at distances a_1, a_2, etc., there are concentrated loads P_1, P_2, etc. There is also a uniformly distributed load of w per unit length. There may be any number of vertical loads and reactions to the left of the origin, but all the vertical loads may be replaced by their resultant Q at some definite distance b from the origin, and all the vertical reactions by a single reaction R

at a distance c from the origin. Writing the moment with respect to a section at a distance x from the origin,

$$M = R(c + x) - Q(b + x) - P_1(x - a_1) - P_2(x - a_2) - \frac{wx^2}{2}; \quad (1)$$

$$M = Rc - Qb + (R - Q)x - P_1(x - a_1) - P_2(x - a_2) - \frac{wx^2}{2}. \quad (2)$$

$Rc - Qb$ is the moment at the origin, which we will represent by M_0, and $R - Q$ is the shear at the origin, which we will represent by V_0.

$$M = M_0 + V_0x - P_1(x - a_1) - P_2(x - a_2) - \frac{wx^2}{2}; \quad (3)$$

$$M = M_0 + V_0x - \Sigma P(x - a) - \frac{wx^2}{2}, \qquad \text{Formula IX.}$$

where $\Sigma P(x - a)$ represents the sum of the moments of all the concentrated loads between the origin and the section considered.

Example

A cantilever of length l (Fig. 81) and weight w per unit length, is fixed at the left end. Find the moment at a distance x from the fixed end by means of the general moment equation.

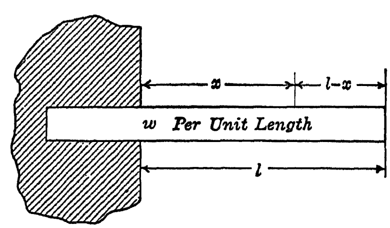


FIG. 81.—Cantilever fixed at left end.

At the origin, by taking the entire beam as a free body it is found

$$M_0 = -\frac{wl^2}{2}, \quad V_0 = wl;$$

$$M = -\frac{wl^2}{2} + wlx - \frac{wx^2}{2} = -\frac{w}{2}(l - x)^2.$$

As a check, consider the portion of length $l - x$ and weight $w(l - x)$ to the right of the section as the free body.

Problems

1. A cantilever of length l and weight w per unit length, is fixed at the left end and carries a load P at a distance a from the fixed end. By the

general moment equation write the expression for the moment for the part between the fixed end and the concentrated load and also for the part between the load and the free end.

2. A beam of length $l + a$ is supported at a distance a from the left end and at right end and carries a load w_1 per unit length from the left end to the first support and a load w_2 per unit length between the supports. With the origin of coördinates infinitely close to the left support, write the moment equation for the part of the beam between the supports.

56. Relation of Moment and Shear.—Differentiate the general moment equation with respect to x.

$$M = M_0 + V_0x - P_1(x - a_1) - P_2(x - a_2) - \frac{wx^2}{2}. \tag{1}$$

$$\frac{dM}{dx} = V_0 - P_1 - P_2 - wx. \tag{2}$$

The right member of (2) is recognized as the shear at a distance x from the origin.

$$\frac{dM}{dx} = V. \qquad \text{Formula X.}$$

The derivative, with respect to the length, of the moment equation of a beam gives the shear in the beam.

From Fig. 79 it is seen that at a concentrated load or at a support there is an abrupt change in the slope of the moment curve. At the concentrated load, Fig. 79, the shear changes from 120 pounds to 20 pounds and there is an equivalent relative change in the slope of the tangent to the moment curve. The shear at this point may be said to have any value between 120 pounds and 20 pounds. The derivative of the moment is not *single-valued* and Formula X does not hold. It does hold, however, infinitely close to this point on either side.

In reality, no load can be concentrated at a point or on a line running across the beam. A so-called concentrated load is distributed over an area, and if we knew the distribution we could express the shear at any point with a single value and Formula X would be found valid at all points.

Since

$$V = \frac{dM}{dx},$$

$$\int V dx = \int dM;$$

$$\int V dx = M_2 - M_1. \qquad \text{Formula XI.}$$

The integral of $V dx$ between any two values of x gives the difference of the moments at the corresponding points.

In Fig. 82 it is seen that Vdx is an element of area between the X axis and the shear diagram. The integral of Vdx between the limits x_1 and x_2 represents the area bounded by the shear diagram, the X axis, and the ordinates at x_1 and x_2. *The area of the shear diagram between two points is the difference between the moments at these points.* Where the shear is negative so that the area is below the X axis the area is regarded as negative. At the ends of a beam, if all the loads and reactions are perpendicular to the length of the beam, the moment is zero; so that the entire area

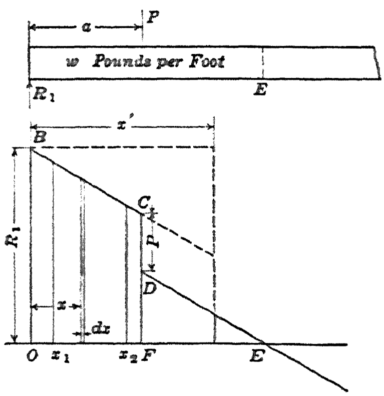


Fig. 82.—Relation of area of shear diagram to moment.

of the shear diagram from the end of the beam to any given point gives the moment at that point.

In Fig. 79 the moment at 2 feet from the left end is the area of the trapezoid $OBCF$ which is, $\frac{160 + 120}{2} \times 2 = 280$ foot-pounds. At 3 feet from the left end it is the moment at 2 feet plus the area of the triangle of 1 foot base and altitude 20 units; so that the moment here is 10 foot-pounds more than at 2 feet. At 4 feet the area of the triangle below the X axis is subtracted from 290 foot-pounds.

Problems

1. A cantilever of length l is fixed at the right end and carries a load P at the left end. Calculate the moment at the right end by means of the shear diagram.

2. A cantilever of length l is fixed at the right end and carries a load of w per unit length, uniformly distributed. Find the moment at the right end by means of the area of the shear diagram.

The area of the shear diagram from the extreme left to any pt = the moment at that pt.

3. A beam of length l is supported at the ends and carries a load P at the middle. Find the moment at one-fourth the length from the left end, and also at three-fourths the length from the left end by means of the shear diagram.

4. In Fig. 79, knowing the moment at 3 feet to be 290 foot-pounds, find the moment at 5 feet, 7 feet, 10 feet and 12 feet by means of the area of the shear diagram.

5. In Fig. 82 write the expression for the moment at a distance x' from the left end, where x' is greater than a, and show what areas on the figure represent the terms of the result.

6. A horizontal cantilever, fixed at the right end, carries a load which increases uniformly from left to the right and is u pounds per unit length at unit distance from the left end. Show that the shear is $-\frac{ux^2}{2}$ and the moment is $-\frac{ux^3}{6}$ at a distance x from the free end.

57. The Dangerous Section.—A section in a beam where the moment has a maximum numerical value is called a *dangerous section.* The mathematical condition for a maximum or minimum value of M is that the derivative with respect to the length shall be zero. But since $\frac{dM}{dx}$ is the shear, this means that there is a dangerous section at every point where the shear becomes zero. In Fig. 79 the shear diagram crosses the X axis at 3 feet from the left end. This is one dangerous section.

The shear may pass through zero when the moment equation does not fulfill the mathematical condition that the slope of the tangent to the curve is zero. At the right support in Fig. 79, the slope of the moment curve changes abruptly from negative to positive. This is a dangerous section as may be seen from the fact that we now add a positive shear area to the negative moment so that the moment begins to decrease numerically.

When the loading of a beam is given, always find the dangerous sections by means of a sketch of the shear diagram. If the dangerous section does not come at a support or under a concentrated head, its exact position may be found algebraically.

Problems

1. A beam 10 feet long, supported at the ends, carries a distributed load of 40 pounds per foot, and a load of 120 pounds 4 feet from the left support. Find the reactions, construct a sketch of the shear diagram, with the values of the shear at the important points, as in Fig. 79, and find the dangerous section. Find the moment at the dangerous section by means of the area of the shear diagram.

Ans. Maximum moment, 768 foot-pounds.

2. In Problem 1 find the moment of the middle by means of the area of the shear diagram to the right of the middle.

3. A beam 12 feet long supported at the ends, carries a load of 30 pounds per foot and a load of 120 pounds 3 feet from the left end. Find the moment at the dangerous section by means of the reactions and loads. With this moment known, find the moment under the concentrated load and at the middle by means of the shear diagram.

4. A beam is supported at the ends and carries a load P at one-third the length from the left end and an equal load at one-third the length from the right end. Draw the shear and moment diagrams, neglecting the weight of the beam.

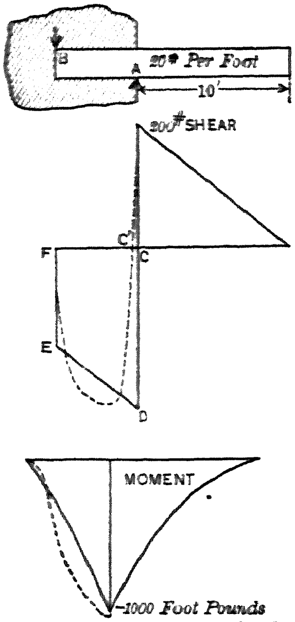


FIG. 83.—Cantilever fixed at left end.

Fig. 83 shows the moment and shear diagrams for a cantilever. If there are no horizontal forces, the area of the shear diagram inside the wall must equal the area outside. The form of the diagram inside the wall is not known. If all the downward forces were concentrated at B, and all the upward force at A, the shear diagram would be the figure $CDEF$. Since the pressures are distributed, the shear diagram is that shown by the dotted lines, and the dangerous section is at C', a little back of the face of the wall.

The actual moment diagram inside the wall is something like that shown by the dotted lines.

CHAPTER VII

STRESSES IN BEAMS

58. Distribution of Stress.—At any section of a bent beam, there is tension across one part of the section and compression across the other, and there is usually shear parallel to the section. The method of finding the total vertical shear has been given in Chapter VI. There remains the problem of finding the tension and compression and the unit tensile and compressive stresses.

If the external forces have no components parallel to the length of the beam, the resultant compressive stress across any section is equal to the resultant tensile stress, and these two forces form a couple, the moment of which is equal to the product of either force multiplied by the distance between them. This moment is equal and opposite to the bending moment.

To calculate these forces (H and C of Figs. 64 to 69) it is only necessary to know the bending moment and the distance between the forces. This distance is easily measured in Figs. 64 and 66. In Fig. 67 the compressive stress is distributed over the small block, and its law of distribution must be known in order to locate its resultant. In Fig. 69, the tensile stress is distributed over the entire upper portion and the compressive stress over the entire lower portion, so that the law of distribution of both must be known in order to find the moment arm.

In a bent beam, the fibers on the convex side are elongated and those on the concave side shortened. Between these there is a surface in which the fibers suffer no deformation in the direction of the length. This surface is called the *neutral surface* of the beam. The intersection of the neutral surface with any transverse section of the beam is the *neutral axis* of that section.

It is customary to assume that the unit stress at any section varies directly as the distance from the neutral axis. The reasons for this assumption and the conditions under which it is valid, will be given in Article 63. Fig. 84 represents graphically the variation of unit stress in a beam, the upper part of which is in tension.

Fig. 84, I, shows the forces from left to right and also the forces from right to left. Fig. 84, II, shows only the forces with which the portion of the beam to the right of the section acts on the portion to the left. It will be noticed that both sets of forces tend to turn the left portion clockwise about the neutral axis at O. Fig. 84, III, shows a convenient method of drawing the diagram to show the magnitude of the unit stress at any distance from the neutral axis.

Since the unit stress varies as the distance from the neutral axis, it may be represented by two wedges cut from the beam by two planes which pass through the neutral axis. One of these planes should be normal to the length of the beam, and, therefore,

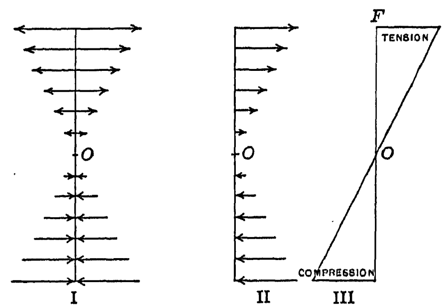


FIG. 84.—Stress variation in a beam.

represent the section considered, and the other may make any convenient angle. Fig. 84, III, may be considered as representing two such planes. The volume of each wedge may be regarded as giving the total stress across its corresponding part of the section, and the distance between the center of gravity of the two wedges, measured parallel to the section, gives the moment arm of these total stresses.

59. Fiber Stress in a Beam of Rectangular Section.—Fig. 85 shows the wedges representing the stress distribution of a rectangular beam section of breadth b and depth d. Since the total tension H is equal to the total compression C, and since the breadth and slope of the wedges is the same, the height must be equal and the neutral axis BB is at a distance $\frac{d}{2}$ from the top or bottom. If S is the unit tensile stress in the top fibers of the beam, the total tension in the upper half is given by

$$H = \frac{bdS}{4}.$$

The center of gravity of a triangle, and consequently of a rectangular wedge, is two-thirds the height from the vertex, so that the distance from H to the neutral axis is $\frac{2}{3} \times \frac{d}{2} = \frac{d}{3}$.

In like manner the resultant compression C is the same distance below the neutral axis, and the total moment arm is $\frac{2}{3}d$.

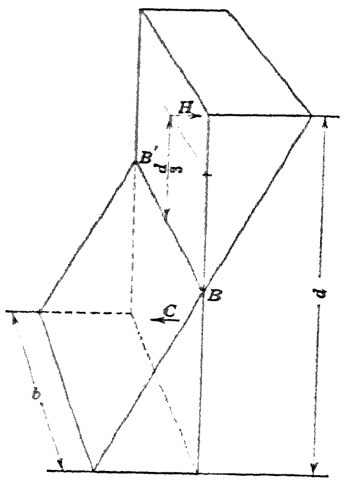


FIG. 85.—Solids representing stress in a rectangular beam.

M = Resisting moment $= \frac{bdS}{4} \times \frac{2\,d}{3} = \frac{Sbd^2}{6}$. Formula XII.

Example

A 4-inch by 6-inch cantilever carries a load of 240 pounds on the free end. Find the unit stress in the top and bottom fibers at a section 5 feet from the free end.

Horizontal dimensions are given first. A 4-inch by 6-inch beam is 4 inches wide and 6 inches deep. Since unit stresses are required in pounds per square inch the moment must be in inch-pounds.

$$M = \frac{Sbd^2}{6},$$

$$S = \frac{6\,M}{bd^2} = \frac{6 \times 14{,}400}{4 \times 36} = 600 \text{ pounds per square inch.}$$

Problems

1. A 6-inch by 10-inch beam, 15 feet long, is supported at the ends and carries a load of 120 pounds per foot, uniformly distributed. Find the maximum unit fiber stress. *Ans.* 405 pounds per square inch.

2. In Problem 1 what is the total tension in the lower half of the beam at the dangerous section? *Ans.* 6,075 pounds.

3. An 8-inch by 12-inch beam 20 feet long, is supported at the ends and carries a load of 160 pounds per foot, uniformly distributed, and a load of 1,280 pounds 5 feet from the left support. Find the maximum fiber stress at the dangerous section, using the portion to the right of the section as the free body. *Ans.* 720 pounds per square inch.

4. Solve Problem 3 if the load of 1,280 pounds is put 7 feet from the left support.

Look up in handbook the working stresses for timber recommended by the American Railway Engineering Association, and state whether Norway pine may be used for this beam.

60. Fiber Stress in a Beam of any Section.—The methods of Article 59 are not convenient to apply to sections other than rectangles. There is a general method which applies to sections of any form, in which the principal terms involve the center of gravity and moment of inertia of plane figures. As these are given in handbooks, a great saving in labor is gained.

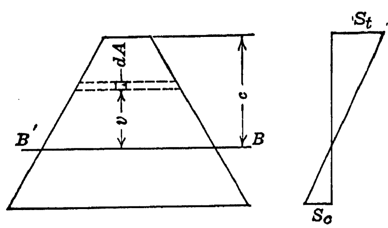


FIG. 86.—Beam section.

Fig. 86 may be regarded as representing a section of any form. BB' is the neutral axis. An element of area dA is at a distance v from the neutral axis. (The letter v will be used to represent distance from the neutral axis in a section, and y will be reserved to represent deflection of the axis from its original position.) The area dA may be infinitesimal in two dimensions or it may extend entirely across the section parallel to the neutral axis as shown by the dotted lines.

Since the unit stress varies as v it may be represented by kv where k is a constant.

On an element of area dA,

$$\text{Total stress} = kvdA. \qquad (1)$$

The moment of this stress on dA about the neutral axis,

$$dM = kv^2dA. \qquad (2)$$

Since v^2 is positive when v is positive or negative, the sign of increment of moment is the same whether the element is above or below the neutral axis.

$$M = k\int v^2dA = kI, \qquad (3)$$

where I is the moment of inertia of the section with respect to the neutral axis. Since

$$s = kv, \quad k = \frac{s}{v}. \tag{4}$$

which substituted in (3) gives

$$M = \frac{sI}{v}. \qquad \text{Formula XIII.}$$

Formula XIII gives the unit stress at any distance from the neutral axis. The most important stress is the stress in the extreme outer fibers where v is a maximum and the unit stress is the greatest. If this maximum unit stress be represented by S and the distance to the outer fiber from the neutral axis be represented by c, the formula becomes

$$M = \frac{SI}{c},$$

This formula is so important that it is desirable to memorize it also in the form

$$S = \frac{Mc}{I}. \qquad \text{Formula XIV.}$$

61. Location of the Neutral Axis.—The values of I and c in Formula XIII depend upon the location of the neutral axis. This is found from the condition that the total tensile stress across the part of the section on one side of the neutral axis is equal to the total compressive stress across the part of the section on the other side of the axis. On an element dA,

$$\text{total stress} = kvdA. \tag{1}$$

$$\text{Total stress on entire section} = k\int vdA = 0. \tag{2}$$

The constant k is not zero when the beam is bent, consequently $\int vdA$ must be zero.

The center of gravity of a plane area is given by

$$\bar{v} = \int \frac{vdA}{A}; \tag{3}$$

$$\bar{v}A = \int vdA = 0. \tag{4}$$

Since A is not zero

$$\bar{v} = 0. \tag{5}$$

The neutral axis of a beam of any section passes through the center of gravity of the section.

[illegible] the outside or neutral axis

we will have 2 values [illegible]

62. Section Modulus.—The expression $\frac{I}{c}$, where c is the distance from the neutral axis to the extreme outer fiber, is called the *section modulus*, or *modulus* of the *section*. Formula XIV becomes

$$S = \text{unit stress in extreme fibers} = \frac{\text{bending moment}}{\text{section modulus}}$$

The values of the section moduli of rolled shapes are given in the handbooks of the steel companies. They are also given for the principal geometric sections (see "Elements of Sections" in Carnegie, and "Properties of Various Sections" in Cambria). In Carnegie, the section modulus is represented by S.

In a rectangular section, $I = \frac{bd^3}{12}$, and $c = \frac{d}{2}$, so that the section modulus of a rectangular section is $\frac{bd^2}{6}$. When this is substituted in Formula XIII it gives

$$M = \frac{Sbd^2}{6}. \qquad \text{Formula XII.}$$

Problems

1. From the handbook find the section modulus of a 12-inch 35-pound I-beam for the axis perpendicular to the web. Check the figure by dividing the moment of inertia given in the table by the distance from the neutral axis to the extreme fiber.

2. Look up the moment of inertia and the location of the center of gravity for a 6-inch by 6-inch by 1-inch angle section. From these calculate the section modulus and compare with the table.

3. Solve Problem 2 for a semicircular area with the neutral axis parallel to the diameter which bounds it.

4. A 15-inch 42-pound I-beam, 25 feet long, is supported at the ends and carries a load of 600 pounds per foot. Find the maximum bending stress at the dangerous section due to this load.

Ans. 9,550 pounds per square inch.

5. Select an I-beam for a span of 30 feet to carry a distributed load, including its own weight, of 1,800 pounds per foot with a maximum unit stress of 15,000 pounds per square inch.

Ans. M = 202,500 foot-pounds = 2,430,000 inch-pounds, section modulus = 162 inches³.

Use a 24-inch 80-pound I-beam having a section modulus of 173.9 inches³.

6. Find the I-beam for a span of 20 feet to carry a load of 7,500 pounds 4 feet from one end and a uniform load, including its own weight, of 900 pounds per foot, with a maximum fiber stress at the dangerous section of 15,000 pounds per square inch. Check moments at dangerous section by taking each end separately as the free body. *Ans.* 15-inch, 42-pound I-beam.

7. Find the total safe load, uniformly distributed, on a 6-inch by 10-inch white-oak beam 15 feet long, supported at the ends, using the allowable unit stress recommended by the American Railway Engineering Association (see handbook).

8. What should be the depth of a beam of Douglas fir, 8 inches wide, for a span of 15 feet, to carry a load of 660 pounds per foot including its own weight?

9. Determine the moment of inertia of a circular section of radius a, and show that the section modulus is $\frac{\pi a^3}{4}$.

10. What is the section modulus of a 6-inch square with the diagonal vertical? How does it compare with the section modulus of the same section with one side vertical? *Ans.* Section modulus, 25.45; ratio, $1 : \sqrt{2}$

11. A square section with diagonal vertical has its section modulus increased by chamfering the top and bottom corners. What must be the dimensions of the triangular sections cut away, in terms of the side of the square, to make the section modulus a maximum?

Ans. One-ninth of the side.

63. Relation of Stress to Deformation.—Fig. 87 represents a bent beam with the concave side upward (the amount of bending

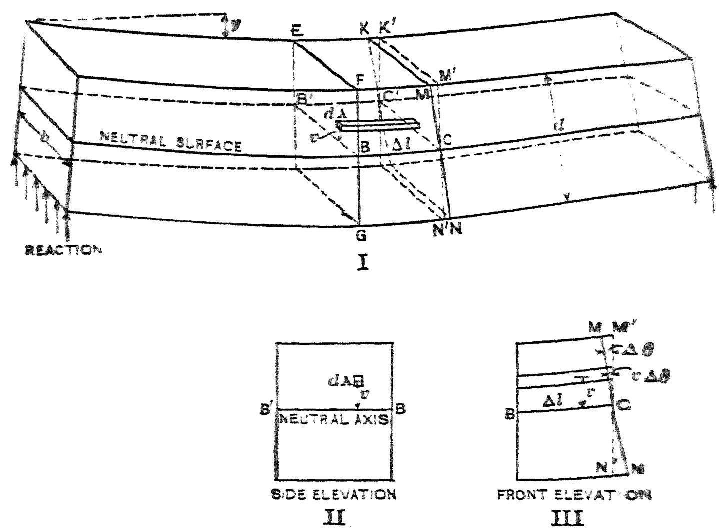


FIG. 87.—Deformation of a bent beam.

is exaggerated). *EFG* is a plane section with neutral axis *BB'*. The dotted lines *K'M'*, *M'N'* indicate the position, before the beam was bent, of a plane section parallel to *EFG* at a distance Δl therefrom. Consider the section *EFG* as fixed and the parts of the beam on both sides of it bent upward. The plane *K'M'N'*

is rotated about its neutral axis CC' through an angle $\Delta\theta$. (There is also a slight shift upward, but this does not affect the problem.) Since there is no elongation in the neutral surface, the distance between the neutral axes BB' and CC', measured along the curved surface, remains unchanged.

It is assumed that a plane section in a beam remains plane when the beam is bent, so that the section becomes the plane KMN. A filament of cross-section dA extends from the plane EFG to the plane KMN, at a distance v from the neutral surface. When the beam is bent, and the plane KMN is turned through the angle $\Delta\theta$, this filament is shortened an amount $v\Delta\theta$. A similar filament below the neutral surface will be elongated $v\Delta\theta$. The unit deformation of the filament is given by

$$\delta = \frac{v\Delta\theta}{\Delta l} \tag{1}$$

Under the condition that the deformations are such that no stress exceeds the proportional elastic limit, the unit stress varies as the unit deformation, and since the unit deformation varies as v, the unit stress varies as the distance from the neutral axis, as was assumed in Article 58.

Since unit stress is equal to $E\delta$, the unit stress above the neutral surface is given by

$$s_c = E_c v \frac{\Delta\theta}{\Delta l}. \tag{2}$$

Below the neutral surface

$$s_t = E_t v \frac{\Delta\theta}{\Delta l}. \tag{3}$$

In most cases **it is assumed** (and is practically true), **that the modulus of elasticity is the same in both compression and tension.** With this assumption,

$$s = Ev\frac{\Delta\theta}{\Delta l}, \tag{4}$$

is the expression for the unit stress in the beam, at any element of area.

On an element of area,

$$\text{Total stress on } dA = Ev\frac{\Delta\theta}{\Delta l}\,dA. \tag{5}$$

The moment of this stress with respect to the neutral axis BB'

is the product of the total stress on dA by the moment arm v; moment of stress about axis,

$$dM = Ev^2\frac{\Delta\theta}{\Delta l}dA = E\frac{\Delta\theta}{\Delta l}v^2dA. \quad (6)$$

The total moment of all the filaments which make up the beam is the integral of dM over the section EFG. Integrating over this area, $\frac{\Delta\theta}{\Delta l}$ remains constant and

$$M = E\frac{\Delta\theta}{\Delta l}\int_{c_1}^{c_2} v^2dA = E\frac{\Delta\theta}{\Delta l}I, \quad (7)$$

where c_1, c_2 are the distances of the lower and upper surfaces of the beam from the neutral surface, I is the moment of inertia of the cross-section EFG or KMN with respect to its neutral axis, and $\Delta\theta$ is the change in slope of the normal to the beam, or the change in slope of the tangent to the beam, in the length Δl.

Problems

1. A 6-inch by 2-inch beam is bent so that two transverse sections, which were originally parallel and 20 inches apart, now make an angle of 1 degree with each other. If the moment throughout the 20-inch length is constant and E is 1,350,000, find the moment.

Ans. $M = 1{,}500\,\pi = 4{,}712.4$ inch-pounds.

2. In Problem 1 find the stress in the outer fibers by Formula XIV and check by equation (4).

3. Through what angle may a steel rod ¼ inch thick be bent in a length of 1 foot, if the maximum fiber stress does not exceed 15,000 pounds per square inch?

To get some idea of the magnitude of the quantities involved, consider Fig. 88. This represents a beam 6 inches wide, 8 inches deep, and about 7 feet long, supported at two points about 80 inches apart. An extensometer (not shown) is attached at two points, F and M, 40 inches apart and 1 inch below the top of the beam. A second extensometer is attached at G and N, 1 inch from the bottom. Two loads of 4,000 pounds each are applied 16 inches from the supports. If the beam is made of timber, the deflection at the middle is about 0.08 inch. (This deflection is too small to show in the drawing unless the scale is exaggerated.) The upper extensometer shows a shortening of about 0.0180 inch in the original length of 40 inches, and the lower extensometer shows an equal elongation. If the tension and compression are exactly equal, the neutral surface is midway between

the extensometers. If the readings are unequal, the location of the neutral surface may be found from the similar triangles, such as MCM', NCN' (Fig. 87), with MM' and NN' known from the extensometer readings, and the distance between the instruments equal to MN. In case the readings are each 0.0180 inch, showing that the neutral axis is at the middle of the section, 4 inches from the top, the compression in the top fibers is four-thirds as great as at M. The compression at 1 inch from the neutral surface is 0.0060 inch; and at a distance v it is 0.0060 v. The unit deformation at a distance v from the neutral axis is 0.00015 v.

It will be noticed that measurements are taken along the chord instead of along the arc, so that the readings are in error the amount of this difference. It may be shown, however, that the error is beyond the limits of the extensometer readings, and, therefore, makes no difference in the result.

Problems

4. A beam is tested as shown in Fig. 88. The points F and M are 40 inches apart and 6 inches above the similar points G and N. The compres-

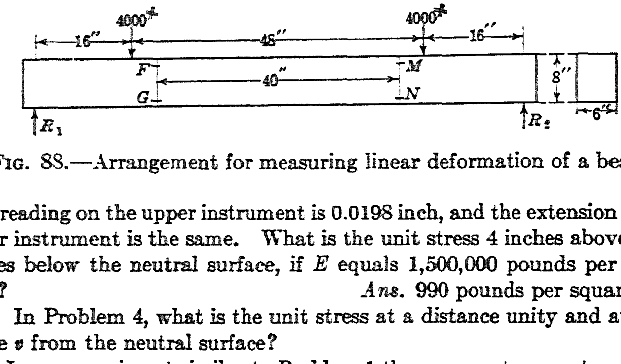


FIG. 88.—Arrangement for measuring linear deformation of a beam.

sion reading on the upper instrument is 0.0198 inch, and the extension on the lower instrument is the same. What is the unit stress 4 inches above and 4 inches below the neutral surface, if E equals 1,500,000 pounds per square inch? *Ans.* 990 pounds per square inch.

5. In Problem 4, what is the unit stress at a distance unity and at a distance v from the neutral surface?

6. In an experiment similar to Problem 1 the upper extensometer shows a shortening of 0.0180 inch, and the lower extensometer an elongation of 0.0220 inch. The beam is 10 inches deep and the extensometers are 1 inch below the top and 1 inch above the bottom, respectively. How far is the neutral axis from the top of the beam? *Ans.* 4.6 inches.

64. Graphic Representation of Stress Distribution.—The unit stress in a beam, provided it does not exceed the proportional elastic limit, varies as the distance from the neutral axis. It may be represented by the straight line GE, Fig. 89, I. This straight line is really a part of the stress-strain diagram for the material in both tension and compression, with the vertical line DF as the

X axis. If the unit stress is carried beyond the elastic limit, it may be represented by the line $KG'E'L$, which is also a stress-strain diagram with one scale changed.

In a beam of rectangular section, the total stress on any area dA, extending across the section, is proportional to the unit stress. The shaded area of Fig. 89, I, may represent the total stress in a rectangular section as well as the unit stress in a section of any form. It is often convenient to represent total stress in a rectangular section by a figure similar to the shaded area in Fig. 89, II. This is really the same as Fig. 89, I, with oblique

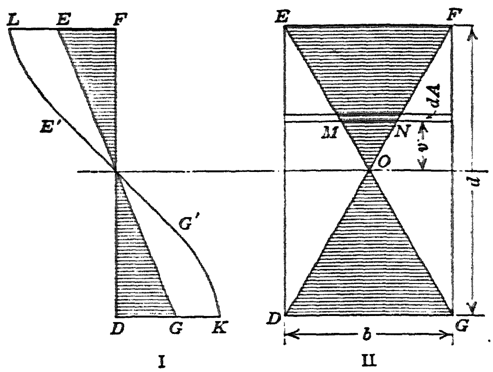


FIG. 89.—Stress distribution in a rectangular section.

axes. The line EF represents the breadth of the section and also the total stress in the extreme outer fibers. It is evident, from the similar triangles, that the total stress on the area dA, extending across the section at a distance v from the neutral axis, will be to the total stress at the top, as the length MN is to the length EF. The actual stress over the entire section is equal to a uniform stress of intensity equivalent to that in the outer fibers over the shaded area. If the cross-section is drawn to full scale, the area of the shaded triangle OEF gives the total stress above the neutral surface when the maximum stress is 1 pound per square inch. Likewise, the area OGD gives the total stress below the neutral surface. These triangular areas are equal in magnitude and opposite in sign, making the sum of the total stress zero.

The shaded triangles of Fig. 89, II, may be regarded as solids of uniform thickness. Fig. 90, I, represents the distribution of stress in the same rectangular section by the method used in

Article 59, but with both tension and compression on one side of the vertical plane which represents the section of the beam. It will be noticed that in Fig. 90, II, the width of the wedge varies

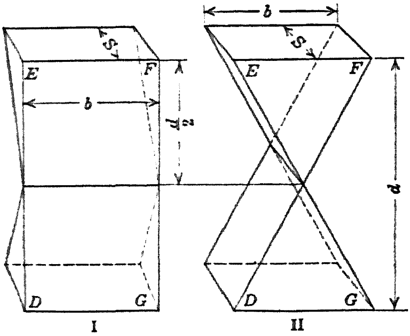


FIG. 90.—Stress distribution solids for a rectangular section.

as the distance from the neutral axis while the thickness, S, is constant; while in Fig. 90, I, the width is constant and equal to that of the section while the thickness varies as the distance from

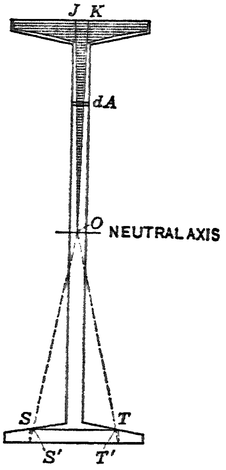


FIG. 91.—Stress distribution in an I-beam.

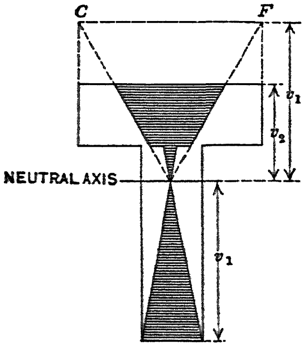


FIG. 92.—Distribution in a T-section.

the neutral axis. From either figure the volume of one wedge is $\frac{Sbd}{4}$ and its center of gravity, which is the location of the resultant force in that half of the section, is $\frac{d}{3}$ from the neutral axis. The

moment of each wedge about the neutral axis is $\frac{Sbd^2}{12}$, and that of the two wedges representing tension and compression in a section is $\frac{Sbd^2}{6}$. Fig. 90, I, shows the actual distribution of stress in a section. It may be called the *stress-distribution solid.* The shaded area of Fig. 90, II, represents a portion of the area of a section on which, if a uniform stress be applied equal to the unit stress in the outer fibers, the total stress and the moment with respect to the neutral axis will be the same as that of the actual distribution. This is called the *stress-distribution diagram*, or modulus figure.*

Fig. 91 is the stress-distribution diagram for an I-beam section. For the rectangular part of the flange this diagram is drawn like Fig. 89. For a small area dA in the web, the length of dA is projected to the top of the section. The ends of the projection are the points J and K. Straight lines are drawn from O to J and K respectively. The part of dA between these lines represents the total stress.

To get the stress on the triangular portion of the flange, consider the portion ST drawn (for convenience) in the lower flange. Project S and T on the lower line and connect the center O with the points thus found by means of the dotted lines. The portion $S'T'$ between these lines measures the total stress. A number of these lines will give the curved area required.

Fig. 92 is the stress distribution in a T-shaped section. The lower portion is constructed like Fig. 89. The outer fibers at the top are nearer the neutral axis than those at the bottom. Instead of projecting on the upper line of the section, we project on a line CF whose distance from the neutral axis is the same as the lower fibers. The total stress in this diagram is expressed in terms of the unit stress in the bottom fibers. We might express the total stress in terms of the unit stress in the top fibers. In that case the lower part of the diagram would extend beyond the section to the right and left.

Problems

1. In a T-section similar to Fig. 92 the flange is 6 inches wide, the total depth is 5 inches, and both flange and stem are 1 inch thick. Find the neutral axis and construct the distribution diagram in terms of the unit stress in the extreme fibers, also in terms of the unit stress at the top of the flange.

* GOODMAN'S "Mechanics Applied to Engineering," uses this name, and gives the figures for a great variety of sections.

2. Construct the distribution diagram for 6-inch by 4-inch by 1-inch angle section, using neutral axis parallel to shorter leg.

65. Stress Beyond the Elastic Limit.—In Fig. 93 the shaded area shows the distribution of stress in a rectangular section, when the stress is considerably beyond the elastic limit. The actual stress in the outer fibers is less than it would be if the modulus were constant in the ratio of the lengths $CH : CF$. The moment of resistance is also less.

Fig. 94 represents the stress distribution when the elastic limit is exceeded as compared with a beam of constant modulus having the same *resisting moment*. The moment of the curved area $OMKHC$ must equal the moment of the triangular area OFC. From the center of the section to the point K the curve lies outside of the straight line. Beyond K it is inside. The unit stress in the

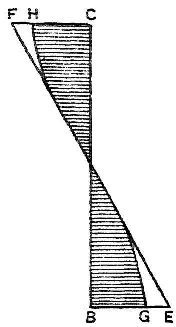


FIG. 93.—Stress distribution diagram beyond the elastic limit.

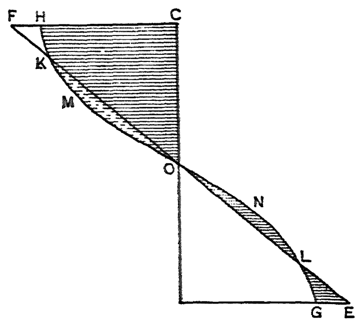


FIG. 94.—Actual and calculated unit stress.

fibers near the neutral surface is greater than if the modulus were constant; and the unit stress in the outer fibers is less. The moment of the dotted area OMK (or the shaded area ONL) is equal to that of the area KFH or LGE.

66. Modulus of Rupture.—When a beam is broken by bending, the stress-distribution diagram for a rectangular section, $OMKH$ (Fig. 94), is similar to the *complete* tension or compression curve of the material. The actual unit stress in the outer fibers is less than that obtained from the equation

$$S = \frac{Mc}{I}. \qquad \text{Formula XIV.}$$

in the ratio of $CH : CF$ (Fig. 94). The *calculated* value of the stress in the outer fibers computed from Formula XIV is called

the *modulus* of *rupture*, or the transverse ultimate strength of the material. It is also called the extreme fiber stress in bending.

Another factor which makes the calculated modulus of rupture different from the actual unit stress, is the shifting of the neutral axis. In sections which are not symmetrical with respect to the axis, the remote fibers on one side reach the elastic limit before those on the other. Fig. 95 represents a T-section. Fig. 95, II, shows part of the stress-strain diagram for both tension and compression. Fig. 95, III, shows the distribution of stress for a small deformation which produces no stress beyond the proportional elastic limit. The neutral axis passes through the center of gravity of the section. Fig. 95, IV, shows the distribution when the deformation is doubled, on the assumption that the neutral axis is not shifted. The lower half of the stem has passed the elastic limit and the unit stress in it is *not* proportional to the distance from the neutral axis. The shaded area below the axis

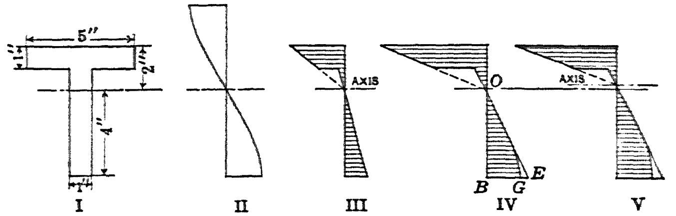


FIG. 95.—Displacement of the neutral axis.

is smaller than that above (which is equal to the triangle OBE) and consequently the neutral axis cannot pass through the point O, but must be moved upward away from the center of gravity of the section. Fig. 95, V, is the actual diagram with the axis shifted. The area above the axis is diminished and that below increased. With a still greater deformation the upper fibers will also pass the elastic limit, and it may happen that, with some forms of stress-strain diagrams, the neutral axis may move backward toward the center of gravity of the section.

The neutral axis is shifted in a *symmetrical* section if the compression and tension curves of the material are not alike. Fig. 96 shows the stress-strain diagrams for cast iron tested at Watertown Arsenal ("Tests of Metals," 1885, pages 475–490). The compression diagram is a straight line up to 13,000 pounds per square inch. The tension diagram curves a little from the start but practically coincides with the compression curve up to

10,000 pounds per square inch. For greater stresses there is a decided difference. Fig. 97, II, shows stress distribution in a rectangular section when the unit tensile stress in the outer

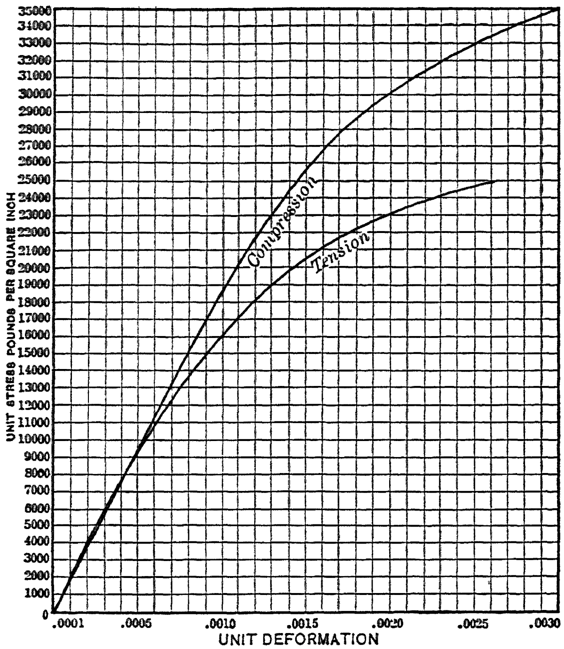


FIG. 96.—Stress-strain diagrams for cast iron.

fibers is 24,800 pounds per square inch, corresponding to the unit elongation of 0.0025. The compression diagram has an equal area when the unit stress is 31,400, which corresponds with the unit deformation of 0.0022. The neutral axis then divides the depth in the ratio of 22 to 25 and is 0.532 d from the top of the beam instead of at the middle. If the resisting moment of the stresses is calculated and substituted in the formula

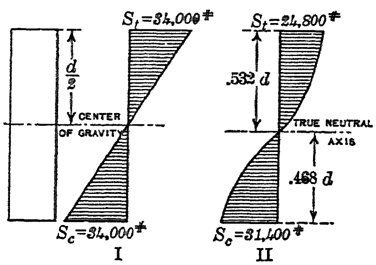


FIG. 97.—Stress distribution in a cast-iron section.

$$S = \frac{Mc}{I},$$

the computed unit stress is 34,000 pounds per square inch. Fig. 97, I, represents the conditions assumed in Formula XIV, while Fig. 97, II, shows the actual conditions.

The compressive strength of cast iron is *three* or *four* times as great as the tensile strength. Beams of this material should be made of T-section or equivalent, and loaded so as to bring the stem in compression and the flange in tension. The remote fibers on the compression side should be *two* or *three* times as far from the center of gravity of the section as those on the tension side.

While the modulus of rupture does not give the actual stress, it enables us to compare stresses in similar sections. If the modulus of rupture is obtained from the test of beams of rectangular section, this figure may be used in computing the ultimate transverse load in other beams of rectangular section made of the same material. The results may also be used with little error for beams of other shapes, provided they are symmetrical with respect to the neutral axis. With unsymmetrical sections, such as angles, it is better to make tests and obtain the modulus of rupture for each shape.

The student will remember, however, that these statements apply to the stress beyond the elastic limit. Since allowable stresses are below the elastic limit, Formula XIV is strictly correct for allowable loads. The change in the stress-distribution diagram when the stress passes the elastic limit *affects* the *factor of safety only*.

Ductile materials, such as soft steel, have no modulus of rupture, strictly speaking, since beams of such material may be bent double without breaking.

Problems

1. A white-pine beam 1.78 inches wide and 1.25 inches thick was supported at two points 12 inches apart and broken by a load at the middle. The total load at rupture was 1,112 pounds. Find the modulus of rupture.

Ans. 7,200 pounds per square inch.

2. If white pine of quality equal to that of Problem 1 is used in the form of a 4-inch by 4-inch beam to carry a load of 700 pounds midway between two supports 6 feet apart, what is the factor of safety?

3. A rectangular bar of cast iron, 1.04 inches wide and 0.80 inch thick is placed on two supports 12 inches apart and broken by a load of 1,635 pounds at the middle. Find the modulus of rupture.

4. A beam of short-leaf yellow pine, tested by Prof. A. N. Talbot at the University of Illinois, had the following dimensions: breadth, 7.12 inches; depth, 16.25 inches; distance between supports, 13 feet 6 inches. Two

equal loads were applied at points 4 feet 6 inches from the supports, making the bending moment constant and the shear zero between these points (if the weight of the beam is neglected). The beam broke by tension in the outer fibers between the loads when each load was 27,500 pounds. Find the modulus of rupture. *Ans.* 4,739 pounds per square inch.

5. A beam of long-leaf yellow pine 7 inches wide and 14 inches deep, supported and loaded as in Problem 4, broke under a *total* load of 37,300 pounds. What was the ultimate bending strength of this timber?

Ans. 4,400 pounds per square inch.

6. A second beam of long-leaf yellow pine 7.0 inches by 12.1 inches, supported and loaded as above, broke under a total load of 52,900 pounds. What was the ultimate bending strength of this timber?

Ans. 8,362 pounds per square inch.

Problems 4, 5 and 6 are from *Bulletin* No. 41 of the University of Illinois Engineering Experiment Station.

7. A 5-inch by 6-inch beam of 1 : 2 : 4 concrete, placed on supports 32 inches apart, was broken by a load of 1,300 pounds midway between the supports. Neglecting the weight of the beam, find the modulus of rupture.

Ans. 347 pounds per square inch.

Prof. G. B. Upton* has devised a method by which the actual unit stress in the outer fibers of a beam of rectangular section may be calculated with considerable accuracy. Fig. 98 is a curve similar to a stress-strain diagram, in which the abscissas represent the deflections of a beam, and the ordinates the unit stresses as calculated from Formula XIV. To find the actual unit stress corresponding to the calculated unit stress FC, draw a line through C tangent to the curve and find its intercept B on the Y axis. One-third the length OB subtracted from the calculated unit stress FC gives the actual unit stress FE.

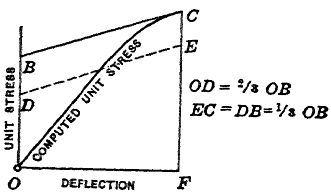


FIG. 98.—Upton's method of finding actual stress in a rectangular beam.

67. Neutral Axis for an Unsymmetrical Section.—Fig. 99 shows two methods of drawing the stress distribution diagram of an angle section. Fig. 99, I, is the usual method with the center O at the middle of the vertical leg on the horizontal line through the center of gravity. Fig. 99, II, shows another method with two centers for the upper portion, each center being directly below the middle of the area to which it corresponds. Both methods give the total stress in terms of the unit stress in the

* For the derivation of this rule see UPTON's "Materials of Construction," page 78.

lower fibers, and both show the distance of the resultant force from a horizontal axis. The center of gravity of the upper shaded area is not directly over that of the lower. Its true position, C of Fig. 99, II, is given by the common center of gravity of the trapezoid and triangle. It corresponds with the center of gravity of the upper part of the stress distribution solid of Fig. 99.

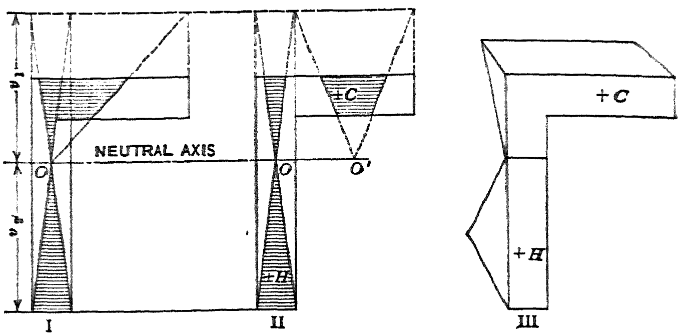


FIG. 99.—Stress distribution in an angle section.

In Fig. 100, $ABCD$ is a rectangular section with diagonal horizontal. It may be regarded as the end of a cantilever perpendicular to the plane of the paper. If a vertical load P is placed on the end of this cantilever, the deflection will not be vertically downward, but the section will be displaced into a position such

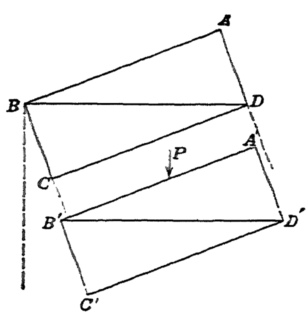


FIG. 100.—Rectangular beam with load perpendicular to diagonal.

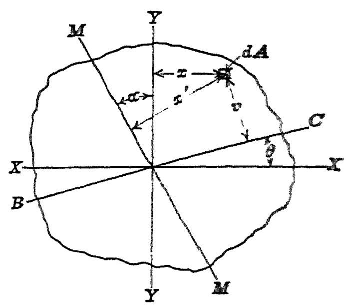


FIG. 101.—Moment at angle with principal axis.

as that of $A'B'C'D'$, from which it is evident that the horizontal diagonal BD is not the neutral axis.

68. Bending Moment Not in Plane of Principal Moment of Inertia.—Figs. 99 and 100 are special examples of the general problem where the bending moment does not lie in the plane

of one of the two principal moments of inertia. Fig. 101 represents any section of a beam for which XX and YY are the principal axes of inertia. The line MM at an angle α with YY is in the plane of the bending moment, and the line BC at an angle θ with XX is the neutral axis. An element dA whose coördinates with respect to the *principal axes are* x and y, and whose distance from the *neutral axis* is v, is subjected to a stress which varies as v.

$$v = y \cos \theta - x \sin \theta. \tag{1}$$

$$s = kv = k\,(y \cos \theta - x \sin \theta). \tag{2}$$

Since the external moment is in the plane MM, the resisting moment must lie in the same plane, so that the sum of the moments of all the stress on the entire area about the line MM must be zero. The perpendicular distance from dA to the line MM is

$$x' = y \sin \alpha + x \cos \alpha. \tag{3}$$

$$\int sx'dA = 0 = k\int vx'dA, \tag{4}$$

$$\int y^2 \cos \theta \sin \alpha \, dA + \int xy \cos \theta \cos \alpha \, dA - \int xy \sin \theta \sin \alpha \, dA - \int x^2 \sin \theta \cos \alpha \, dA = 0; \tag{5}$$

$$I_x \cos \theta \sin \alpha - I_y \sin \theta \cos \alpha = 0, \tag{6}$$

where I_x is the moment of inertia with respect to the axis XX, and I_y is the moment of inertia with respect to YY. The second and third terms of (5) include the product of inertia with respect to the principal axes, which is zero (see Article 207).

$$\operatorname{Tan} \theta = \frac{I_x}{I_y} \tan \alpha. \tag{8}$$

Example

A 6-inch by 8-inch beam is subjected to a load perpendicular to its length making an angle of 30 degrees with the plane of the 8-inch faces. Find the angle between the neutral axis and the planes of the 6-inch faces.

Taking a line through the center parallel to the 6-inch faces as the X axis,

$$I_x = \frac{6 \times 8^3}{12} = 256,$$

$$I_y = \frac{8 \times 6^3}{12} = 144.$$

$$\text{Tan } \theta = \frac{256}{144} \times 0.5774 = 1.0264,$$

$$\theta = 45° 45'.$$

The neutral axis makes an angle of 15 degrees 45 minutes with the line normal to the bending moment.

From Fig. 101 the component of the bending moment perpendicular to the neutral axis is $M \cos (\theta - \alpha)$. The moment of inertia with respect to the neutral axis is $I_x \cos^2 \theta + I_y \sin^2 \theta$, and $v = y \cos \theta - x \sin \theta$.

$$s = \frac{M \cos (\theta - \alpha) v}{I_x \cos^2 \theta + I_y \sin^2 \theta}; \tag{1}$$

$$s = \frac{M (\cos \theta \cos \alpha + \sin \theta \sin \alpha)(y \cos \theta - x \sin \theta)}{I_x \cos^2 \theta + I_y \sin^2 \theta}; \tag{2}$$

$$s = \frac{My (\cos^2 \theta \cos \alpha + \cos \theta \sin \theta \sin \alpha) - Mx (\cos \theta \sin \theta \cos \alpha + \sin^2 \theta \sin \alpha)}{I_x \cos^2 \theta + I_y \sin^2 \theta}. \tag{3}$$

$$My (\cos^2 \theta \cos \alpha + \cos \theta \sin \theta \sin \alpha) = My \cos \alpha (\cos^2 \theta + \cos \theta \sin \theta \tan \alpha) = My \cos \alpha \left(\cos^2 \theta + \frac{I_y}{I_x} \sin^2 \theta\right); \tag{4}$$

$$\frac{My (\cos^2 \theta \cos \alpha + \cos \theta \sin \theta \sin \alpha)}{I_x \cos^2 \theta + I_y \sin^2 \theta} = \frac{My \cos \alpha}{I_x}. \tag{5}$$

In a similar way the second part of (3) may be shown to be

$$\frac{Mx \sin \alpha}{I_y},$$

and

$$s = \frac{My \cos \alpha}{I_x} - \frac{Mx \sin \alpha}{I_y}. \tag{6}$$

To find the fiber stress at any point in a beam when the bending moment is inclined to the principal axes of inertia, resolve the bending moment (or the applied forces) perpendicular to the two axes and compute the stress for each component separately. The actual unit stress is the sum of the results of these two, taken with the proper sign.

Examples

A 6-inch by 8-inch beam 15 feet long is supported at the ends and carries a load of 800 pounds at the middle. The load is 30 degrees from the vertical in a plane normal to the length of the beam. Find the unit stress at each corner at the dangerous section.

The bending moment at the dangerous section is 36,000 inch-pounds

Its components are 36,000 cos 30 degrees vertical, and 36,000 sin 30 degrees. horizontal. The section modulus for the vertical component is 64 inches³ and for the horizontal component it is 48 inches³. Considering first the vertical component of the moment, the unit stress in all the top and bottom fibers is

$$S = \frac{36{,}000 \times 0.866}{64} = 487 \text{ pounds per square inch.}$$

It is compression at the top and tension at the bottom. In the same way the unit stress in the vertical edge AD is 375 pounds per square inch tension,

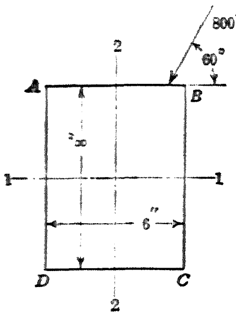


FIG. 102.—Rectangular section with oblique loading.

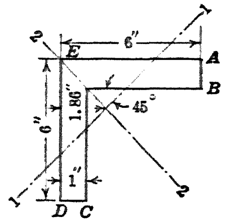


FIG. 103.—Angle section.

and there is an equal unit compressive stress in the vertical edge BC At the corner B both the horizontal and vertical components produce compression so that the actual unit stress is 487 + 375 = 862 pounds per square inch. There is an equal unit tensile stress at the corner D. At corner A the vertical component produces compression and the horizontal component tension, and the actual unit stress is 487 − 375 = 112 pounds per square inch compression. There is an equal tension at corner C.

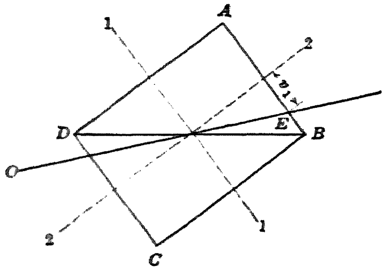


FIG. 104.—Rectangular beam.

A 6-inch by 6-inch by 1-inch standard angle, 10 feet long is used as a beam supported at the ends. The angle is placed with legs horizontal and vertical and a load of 1,000 pounds is applied at the middle, over the center of gravity of the section. Find the unit stress at the corners.

Here the principal axes are 1-1 for which the inertia is 14.78, and 2-2 for which the moment of inertia is 56.14. The bending moment for each axis is 15,000 $\sqrt{2}$.

$$\text{Unit stress at } E = \frac{15{,}000 \times \sqrt{2} \times 1.86 \times \sqrt{2}}{14.78} = 3{,}775 \text{ lb./in.}^2$$

The unit tensile stress at C due to the moment about the axis 1–1 is 3,329 pounds, and the unit tensile stress due to the moment about the axis 2–2 is 1,336, making the total unit tensil stress at this corner 4,655 pounds per square inch.

Problems

1. A rectangle of sides b and d is placed with one diagonal horizontal and subjected to a vertical load. Find the fiber stress at the corners A and C, Fig. 104.

Ans. $s = \dfrac{6M\sqrt{d^2 + b^2}}{b^2d^2}$.

2. Show that the result of Problem 1 is the same as would be found if the neutral axis coincided with the horizontal diagonal DB.

3. In Problem 1 find the unit stress at the corners B and D, and the direction of the neutral axis.

69. Bending Moments in Different Planes.—It frequently happens that a beam is subjected to forces which are not all parallel. If the beam has a circular or square section so that the moment of inertia is the same in all directions, the resultant moment may be calculated at any section and this moment may be used to find the fiber stress. If the two principal moments of inertia are not equal, the forces or moments should be resolved in the directions of the principal axes, and the stress at any point calculated as in Article 68.

Example

A horizontal cantilever 5 feet long carries a load of 120 pounds per foot and is subjected to a horizontal pull, perpendicular to its length, of 400 pounds at the free end. Find the expressions, in inch-pounds, for the moment at any section.

Ans. $M_y = 5x^2$; $M_z = 400x$; Resultant $M = 5x\sqrt{x^2 + 6{,}400}$,

where M_y is the moment in the vertical plane and M_z is the moment in the horizontal plane.

Problems

1. In the example above, find the direction and magnitude of the resultant moment at the fixed end.

Ans. 30,000 inch-pounds in a plane at an angle of 36 degrees 52 minutes with the horizontal.

2. If the cantilever of Problem 1 is of circular section, 4 inches in diameter, find the maximum fiber stress. *Ans.* 4,775 pounds per square inch.

3. In Problem 1 find the magnitude and direction of the resultant moment at 30 inches from the free end.

Ans. 12,816 inch-pounds at angle of 20 degrees 40 minutes with the horizontal.

4. A horizontal shaft 10 feet long, weighing 24 pounds per foot, is sup-

ported at the ends and carries a vertical load of 60 pounds 2 feet from the left end and a vertical load of 40 pounds 3 feet from the right end. A horizontal force of 160 pounds, perpendicular to the shaft, is applied 3 feet from the left end. Find the resultant moment at 3 feet from the left end and at the middle.

Ans. 501 foot-pounds at 3 feet; 484 foot-pounds at 5 feet.

5. Write an expression for the moment in each plane and for the resultant moment between 3 feet and 7 feet from the left support. Differentiate the expression for the resultant moment to derive an equation for finding the position of maximum moment.

Fig. 105, shows the diagrams, for the moment in the vertical and horizontal planes for Problem 4. The maximum moment in the vertical plane is at 5 feet, and the maximum in the horizontal

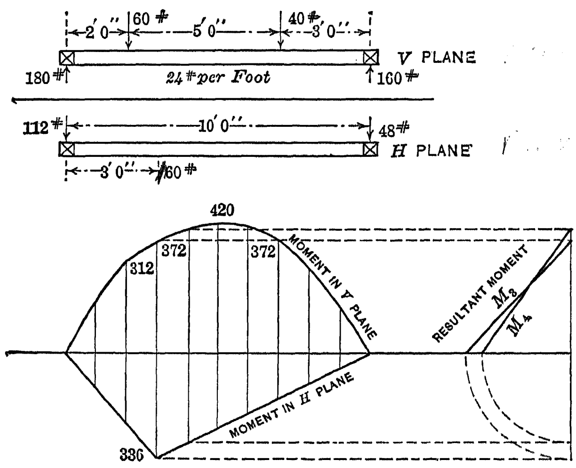


FIG. 105.—Beam with horizontal and vertical loading.

plane is at 3 feet. The maximum resultant moment is between 3 feet and 5 feet. The resultant moment at any section may be determined graphically from the diagonal of the right-angled triangle, the legs of which are the horizontal and vertical moments.

Problems

6. A horizontal shaft 12 feet long, supported at the ends, carries a vertical load of 600 pounds 3 feet from the left end, and a horizontal pull of 400 pounds perpendicular to its length at 6 feet from the left end. Find the location of the maximum resultant moment.

TABLE X.—ALLOWABLE UNIT BENDING STRESSES

Material	Unit stress in pounds per square inch
Structural steel	16,000
Wrought iron	12,000
Cast steel	16,000
Cast iron in tension	3,000
Cast iron in compression	15,000
Concrete in compression:	
1:2:4	650
1:3:6	450
Long-leaf yellow pine	1,200
Douglass fir	1,100
White oak	1,100
Norway pine	900

[illegible] 12' [illegible]
[illegible]
[illegible]
[illegible] 2")

CHAPTER VIII

DEFLECTION IN BEAMS

70. Deflection and Moment.—In Article 63, equation (7), we have:

$$M = EI\frac{\Delta\theta}{\Delta l} = EI\frac{d\theta}{dl}, \tag{1}$$

for infinitestimal lengths dl, measured along the neutral surface of the bent beam. The angle $d\theta$ is the change in slope of the

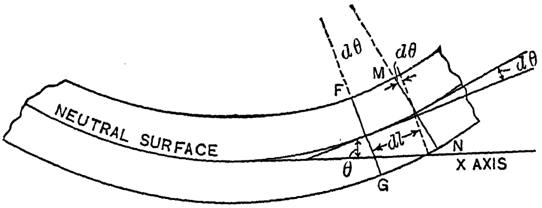


Fig. 106.—Curvature of beam.

tangent to the neutral surface in the length dl. We will now determine the relation existing between the moment and the *deflection* of the beam from its original form. This is expecially easy in polar coördinates. The lines FG and MN, of Figs. 106

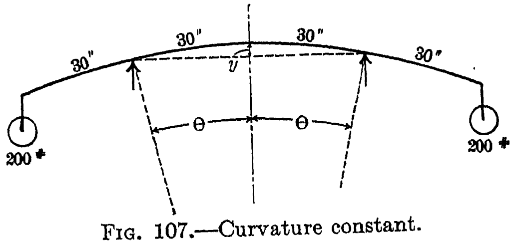


Fig. 107.—Curvature constant.

and 87, make an angle $d\theta$ with each other (when Δl becomes dl), and intersect at some point beyond the drawing, at a distance ρ from the neutral surface. This distance ρ is the radius of curvature of the neutral surface.

By geometry:

$$\rho d\theta = dl, \tag{2}$$

$$\frac{d\theta}{dl} = \frac{1}{\rho}. \tag{3}$$

Substituting in (1),

$$\frac{M}{EI} = \frac{1}{\rho}, \quad M = \frac{EI}{\rho}. \tag{4}$$

If M is constant, or if I varies as M, ρ is constant, and the curve of the beam is an arc of a circle which may be computed by trigonometry.

Problems

1. A 3-inch by 1-inch steel beam, 10 feet long rests, on two supports each 30 inches from the ends and carries a load of 200 pounds on each end. Neglecting the weight of the beam, what is the bending moment at the supports? If the modulus of elasticity is 30,000,000, what is the radius of curvature? How much is the beam deflected upward at the middle?

Ans. Moment, 6,000 inch-pounds; Radius of curvature, 1,250 inches; Deflection at the middle, 0.36 inch.

SUGGESTION.—With the radius of curvature known, calculate the angle in radians subtended by half the span. The versed sine of this angle multiplied by the radius of curvature is the deflection at the middle. As ordinary tables are of little value for such small angles, it is recommended that the student use the first two terms of the cosine series to get this versed sine. (See trigonometry for series or develop by Maclaurin's formula.)

2. A steel plate 1 inch wide and ¼ inch thick is subjected to a bending moment of 125 inch-pounds. If E is 30,000,000 pounds per square inch, what is the radius of curvature and what is the maximum fiber stress?

Ans. Radius of curvature, 312.5 inch; unit stress, 12,000 pounds per square inch.

3. What is the thickness of a saw blade which may be bent into a curve of 5-foot radius with a maximum unit stress of 40,000 pounds per square inch, if E is 30,000,000? *Ans.* 0.16 inch.

71. Deflection in Rectangular Coördinates.—To express the value of $\frac{M}{EI}$ in rectangular coördinates, we must determine $\frac{d\theta}{dl}$ in terms of x and y and their derivatives. Let x express distance parallel to the unbent beam and y express deflection of the beam from its original position. These distances are usually measured along the neutral surface. The angle θ may be measured from any fixed line. For convenience of calculation, we will measure θ from a line parallel to the X axis. The angle θ is then the angle which the tangent to the curved beam makes with the original direction of the beam. It is the angle through which the tangent to the beam at any point is turned when the beam is bent.

Fig. 108 shows a beam supported at the ends and bent. The lower figure represents the neutral axis with the vertical deflection exaggerated. The origin is taken at the left support, and x is taken as positive to the right and y as positive upward, as is the custom in mathematical work.

From Fig. 108 (or the Calculus):

$$\tan\theta = \frac{dy}{dx}. \tag{1}$$

Differentiating

$$\sec^2\theta\,\frac{d\theta}{dx} = \frac{d^2y}{dx^2}. \tag{2}$$

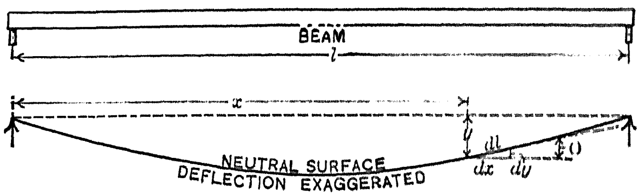


FIG. 108.—Slope and deflection.

From Fig. 108, $dx = dl\cos\theta$, which substituted in (2) gives:

$$\frac{\sec^2\theta\,d\theta}{dl\cos\theta} = \frac{d^2y}{dx^2}; \tag{3}$$

$$\sec^3\theta\,\frac{d\theta}{dl} = \frac{d^2y}{dx^2}; \tag{4}$$

$$\frac{d\theta}{dl} = \frac{d^2y}{\sec^3\theta\,dx^2}. \tag{5}$$

Substituting this value of $\frac{d\theta}{dl}$ in (1) of Article 70:

$$M = \frac{EI}{\sec^3\theta}\,\frac{d^2y}{dx^2} = EI\,\frac{\dfrac{d^2y}{dx^2}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}. \tag{6}$$

Equation (6) may be obtained direct from (4) of Article 70 if the student remembers the expression for the radius of curvature from his Calculus.

In beams, as used in engineering practice, the deflection is usually small, so that $\cos\theta$ is practically unity. Equation (6) then becomes approximately,

$$M = EI\,\frac{d^2y}{dx^2}. \qquad \text{Formula XV.}$$

$M = \frac{SI}{c} = EI\frac{d\theta}{dl} = \frac{EI}{\rho} = EI\,\frac{1}{}$

Formula XV is the differential equation from which the deflection of beams and columns is determined. The X axis is taken parallel to the direction of the unbent beam, and the deflection is relatively small so that $\left(\frac{dy}{dx}\right)^2$ is negligible compared with unity.

72. Point of Inflection.—A point of inflection of a curve is a point where the center of curvature changes from one side to the other. From the Calculus, it is a point where $\frac{d^2y}{dx^2}$ equals zero and changes sign. In a beam $\frac{d^2y}{dx^2} = \frac{M}{EI}$, so that a point of inflection must come at point where the moment is zero. In order to have a change of sign, $\frac{d^3y}{dx^3}$, or some higher derivative of odd order, must *not* be equal to zero.

When $\frac{d^2y}{dx^2}$ equals zero

$$EI\frac{d^3y}{dx^3} = \frac{dM}{dx} = V = \text{vertical shear,}$$

in a uniform beam where I is constant. At a point of inflection the moment is zero, and the shear is usually not zero.

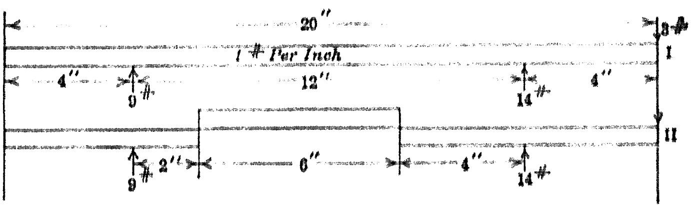


FIG. 109.—Points of inflection.

A point of inflection is sometimes called a point of *counterflexure.* In Fig. 109 there is a point of inflection 2 inches from the left support and another 4 inches from the right support. A beam may be cut in two at a point of inflection and the portions placed on each other so as to take the shear; and the deflections, moments, and reactions, will remain exactly as they were before the beam was cut.

Problems

1. In Fig. 109 calculate the reactions. *Ans.* 9 pounds and 14 pounds.

2. Write the moment equation for the part between the supports with the origin at the left support. Equate to zero and find the two points of inflection. *Ans.* 2 inches and 8 inches from the left support.

3. In Fig. 109, II, take moments about the left support and show that

the left portion exerts a force upon the middle portion, which is equal to half the weight of the latter.

73. Solution of the Differential Equation of Deflection.—Before solving Formula XV for the deflection of a beam or column, all the factors must be expressed in terms of x, y, and *constants.* The modulus of elasticity is constant, provided the unit stress does not exceed the proportional elastic limit. The formulas for deflection are valid only below this limit. For beams of uniform section, I is constant; for beams of variable section, it is expressed as a function of x. The moment is expressed as a function of x and y. In beams it is usually a function of x only as in equations of Article 54.

When the expressions for M and I do not depend upon the deflection y, Formula XV becomes

$$\frac{d^2y}{dx^2} = \text{function of } x. \tag{1}$$

The solution of equation (1) involves the double integration of

$$\frac{d^2y}{dx^2}dx\,dx.$$

Since $\frac{d^2y}{dx^2}$ is the derivative of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}\,dx$ is the differential of $\frac{dy}{dx}$, and

$$\frac{d^2y}{dx^2}dx\,dx = d\left(\frac{dy}{dx}\right)dx = dp\,dx, \tag{2}$$

where

$$p = \frac{dy}{dx}.$$

A double integration may be performed in either of two ways depending upon the order of integration. In $dp\,dx$ either dp or dx may be integrated first. There are two methods of solving (1) corresponding to these two orders of integration. The method in general use is the one in which dp is integrated first. It may be called the *method* of *double integration.* In the other, dx is integrated first. It is called the method of *Area Moments.* Articles 74 to 83 give the application of the first method to beams with two supports. Articles 84 to 93 apply the second method to similar problems. Either method may be studied and the other omitted.

DEFLECTION OF BEAMS WITH TWO SUPPORTS BY METHOD OF DOUBLE INTEGRATION

(*Articles* 74 *to* 83 *may be omitted and Articles* 84 *to* 93 *taken instead.*)

74. The Steps of the Solution.—*The expression*

$$\int\int \frac{d^2y}{dx^2}dx\,dx = \int\int d\left(\frac{dy}{dx}\right)dx = \int\int dp\,dx =$$
$$\int (p + C_1)\,dx = \int\left(\frac{dy}{dx} + C_1\right)dx,$$

when dp is integrated first. C_1 is an integration constant. (The integration might have been taken between limits, one of which should be the variable x, but the use of the integration constant is preferable with this method.)

The second integration gives the deflection y. If the value of $\frac{dy}{dx}$ is known at any point, it may be substituted and the expression for C_1 obtained before the second integration. The value

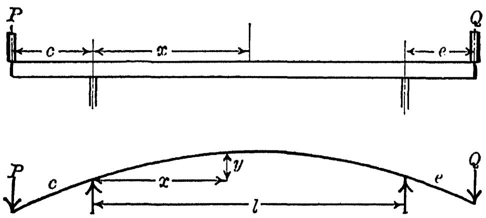


FIG. 110.—Beam with constant moment.

of y must be known for some value of x in order to obtain the constant which comes in at the second integration. If $\frac{dy}{dx}$ is not known for some one value of x, the constant C_1 must be carried through the second integration and y must be known for two values of x. This is the case in Article 75 which follows, while in Article 76, $\frac{dy}{dx}$ is known at one point, and C_1 is determined before the second integration is performed.

75. Beam with Constant Moment.—Fig. 110 represents a beam resting on two supports at a distance l apart. It overhangs the left support a distance c and carries a load P at the left end, and overhangs the right support a distance e and carries a load Q at the right end. The loads are such that $Pc = Qe$.

Taking moments about the right support, the reaction at the left support is found to be equal to P. Taking moments about the left support, the right reaction is found to be equal to Q. Between the supports the moment is constant and equal to $-Pc$.

$$EI\frac{d^2y}{dx^2} = -M, \tag{1}$$

where $M = Pc$. Integrating:

$$EI\frac{dy}{dx} = -Mx + C_1, \tag{2}$$

$$EIy = -\frac{Mx^2}{2} + C_1x + C_2. \tag{3}$$

To obtain the constants C_1 and C_2, we have the condition that $y = 0$ at the left support where $x = 0$. Substituting in equation (3),

$$C_2 = 0;$$

$$EIy = -\frac{Mx^2}{2} + C_1x. \tag{4}$$

Equation (4) is true for all values of x for which the moment is $-M$. It is true at the right support, where $x = l$ and $y = 0$. Substituting in (4):

$$0 = -\frac{Ml^2}{2} + C_1l, \qquad C_1 = \frac{Ml}{2}.$$

Substituting this value of C_1 in equation (4),

$$EIy = -\frac{Mx^2}{2} + \frac{Mlx}{2}; \tag{5}$$

$$y = -\frac{M}{2\,EI}(x^2 - lx) = \frac{M}{2\,EI}(lx - x^2). \tag{6}$$

Equation (6), which gives the value of y in terms of x and the constants EI and M, for all values of x between the supports, is called the *equation* of the *elastic line* of the beam between these points. To find the position of maximum deflection, let $\frac{dy}{dx} = 0$ in equation (2). After substituting the value of C_1,

$$\frac{dy}{dx} = \frac{M}{EI}\left(\frac{l}{2} - x\right),$$

$$\frac{dy}{dx} = 0 \text{ when } x = \frac{l}{2},$$

hence the point of maximum deflection is at the middle of the span. Substituting in (6), to get the maximum deflection at the middle:

$$y_{\max} = \frac{Ml^2}{8\,EI}. \tag{7}$$

It is evident that if the loads P and Q are equal and the lengths c and e are equal, the beam would be symmetrical with respect to the middle of the span; and that this point would be the position of maximum deflection. In that case we could set $\frac{dy}{dx}$ equal to 0 when x equals $\frac{l}{2}$ in equation (2) and solve for C_1 before integrating the second time. If P and Q are not equal, but the products

$$Pc = Qe,$$

the symmetry is not so self-evident, and it is safer to obtain the constants as we have done.

Problems

1. Show that the deflection for all parts of the span is positive.

2. Apply equation (7) to Problem 1 of Article 70.

3. Apply equation (6) to the above problem to find the deflection at 10 inches, 20 inches, and 40 inches from the left support.

Ans. x	y
10 inches.	0.20 inch.
20 inches.	0.32 inch.
40 inches.	0.32 inch.

4. In the above problem find the slope of the tangent at either support, and find how much $1 + \left(\frac{dy}{dx}\right)^2$ differs from unity.

5. A yard stick 1 inch wide and $\frac{1}{4}$ inch thick rests on two supports 20 inches apart. A load of 4 pounds is placed 6 inches to the left of the left support and an equal load 6 inches to the right of the right support. A point mid-way between the supports is elevated 0.64 inch when the loads are applied. Find E. *Ans.* 1,440,000 pounds per square inch.

76. Cantilever Beam with Load on the Free End.—Fig. 111 represents a cantilever beam fixed at the right end and loaded at the left end. The origin of coördinates is taken from the left end before the load was applied. The moment at a distance x from the origin is $-Px$. The differential equation becomes:

$$EI\frac{d^2y}{dx^2} = -Px. \tag{1}$$

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + C_1. \tag{2}$$

At the wall, where $x = l$, the beam is horizontal and $\dfrac{dy}{dx} = 0$;

$$C_1 = \frac{Pl^2}{2}. \tag{3}$$

$$\frac{C_1}{EI} = \frac{Pl^2}{2\,EI}$$

is the slope of the tangent to the elastic line at the origin.

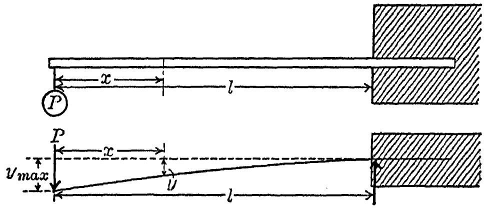


FIG. 111.—Cantilever with load on free end.

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + \frac{Pl^2}{2}. \tag{4}$$

$$EIy = -\frac{Px^3}{6} + \frac{Pl^2x}{2} + C_2. \tag{5}$$

At the wall $x = l$, $y = 0$;

$$0 = -\frac{Pl^3}{6} + \frac{Pl^3}{2} + C_2;$$

$$C_2 = -\frac{Pl^3}{3}. \tag{6}$$

$$EIy = -\frac{Px^3}{6} + \frac{Pl^2x}{2} - \frac{Pl^3}{3}; \tag{7}$$

$$y = -\frac{P}{EI}\left(\frac{x^3}{6} - \frac{l^2x}{2} + \frac{l^3}{3}\right). \tag{8}$$

The maximum deflection is at the free end, where $x = 0$.

$$y_{\max} = -\frac{Pl^3}{3\,EI}. \qquad \text{Formula XVI.}$$

Problems

1. A 4-inch by 6-inch wooden cantilever, 10 feet long, is deflected 1 inch at the end by a load of 200 pounds at the end. Find E and the maximum fiber stress.

Ans. E, 1,600,000 pounds per square inch; maximum stress, 1,000 pounds per square inch.

2. In Problem 1 find the deflection 60 inches from the free end.

Ans. $\frac{5}{16}$ inch.

3. If the deflection at the end of a cantilever due to a load on the end is 1 inch, find the deflection at one-fourth the length from the free end.

4. If E is 1,500,000 and the maximum allowable stress is 1,200 pounds per square inch, what is the maximum deflection in a cantilever 10 feet long and 4 inches deep, if the neutral axis is midway between the top and bottom.

Ans. 1.92 inches.

5. A 12-inch I-beam as a cantilever 8 feet 4 inches long is deflected $\frac{5}{16}$ inch at the end by a load at the end. If E is 29,000,000 pounds per square inch, find the maximum fiber stress. *Ans.* 14,500 pounds per square inch.

6. If l is length of a cantilever and d is the depth, show that the deflection due to a load on the end is given by

$$y_{\max} = \frac{2\,Sl^2}{3\,Ed}$$

7. A cantilever of length $a + b$ carries a load P on the free end. Find the deflection at a distance a from the free end. *Ans.* $y = -\frac{Pb^2}{6\,EI}(3\,a + 2\,b)$.

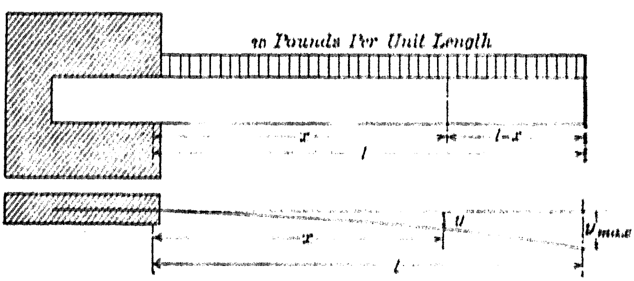


FIG. 112.—Cantilever with uniformly distributed load.

77. Cantilever with Load Uniformly Distributed.—Fig. 112 represents a cantilever fixed at the *left* end, with load uniformly distributed. The origin of coördinates is taken at the fixed point. The moment is determined from a free portion of length $l-x$ to the right of a section which is at a distance x from the origin. The weight of the portion with a distributed load of w per unit length is $w\,(l-x)$. The moment arm with respect to the section is $\frac{l-x}{2}$.

$$M = -\frac{w(l-x)^2}{2}.$$

The sign of the moment is negative, the center of curvature being below the beam.

$$EI\frac{d^2y}{dx^2} = -\frac{w(l-x)^2}{2}. \qquad (1)$$

$$EI\frac{dy}{dx} = \frac{w(l-x)^3}{6} + C_1. \tag{2}$$

At the wall, where $x = 0$, $\frac{dy}{dx} = 0$;

$$0 = \frac{wl^3}{6} + C_1. \tag{3}$$

$$EI\frac{dy}{dx} = \frac{w(l-x)^3}{6} - \frac{wl^3}{6}. \tag{4}$$

$$EIy = -\frac{w(l-x)^4}{24} - \frac{wl^3x}{6} + C_2. \tag{5}$$

At $x = 0$, $y = 0$;

$$C_2 = \frac{wl^4}{24}. \tag{6}$$

$$EIy = -\frac{w(l-x)^4}{24} - \frac{wl^3x}{6} + \frac{wl^4}{24}. \tag{7}$$

The maximum deflection is at the point where $x = l$;

$$y_{\max} = -\frac{wl^4}{8\,EI} = -\frac{Wl^3}{8\,EI}. \qquad \text{Formula XVII.}$$

where $W = wl$, the total distributed load.

The maximum deflection at the right end is not a mathematical maximum where the slope of the tangent is zero. It is a numerical maximum since the curve ends at that point. The curves considered in Calculus are indefinite in extent.

If the beam is fixed at the right end and the origin is taken at the left end, as in the preceding article, $(l-x)$ and x will be interchanged in (7) and

$$EIy = -\frac{wx^4}{24} - \frac{wl^3(l-x)}{6} + \frac{wl^4}{24}. \tag{8}$$

$$EIy = -\frac{wx^4}{24} + \frac{wl^3x}{6} - \frac{wl^4}{8}. \tag{9}$$

Problems

1. What is the deflection at the end of a 4-inch by 6-inch wooden cantilever, 10 feet long, due to a distributed load of 40 pounds per foot if E is 1,200,000 pounds per square inch? What is the maximum fiber stress?

Ans. $y_{\max} = 1$ inch; $S = 1{,}000$ pounds per square inch.

2. A beam of length l and depth d as a cantilever carries a load, uniformly distributed, which makes the maximum unit stress at the dangerous section equal to S. Find the expression for the deflection at the free end.

Ans. $y_{\max} = \frac{Sl^2}{2\,Ed}$.

3. An 8-inch 18-pound I-beam, as a cantilever 5 feet long, carries a distributed load of 120 pounds per foot and a load of 500 pounds at the free end. Find the deflection at the free end and the maximum fiber stress. E equals 29,000,000 pounds per square inch.

4. Expand equation (1) and integrate to get the equation of the elastic line. Then expand equation (7) and compare the results.

5. Find the equation of the elastic line of a cantilever fixed at the right end, with the origin at the left end. Compare the result with equation (9).

78. Deflection from Tangent.—If the tangent line to a cantilever at the "fixed point" is not horizontal, the equations of Articles 76 and 77 give the deflection from this tangent line and

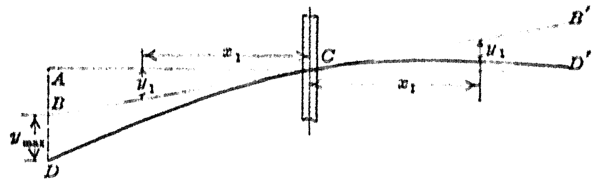


FIG. 113.—Deflection from tangent.

not from the horizontal line. In Fig. 113, DC represents a bent beam, and BC is a straight line tangent to the beam at C. The distance BD is y_{max} of the formulas of Articles 76 and 77.

A beam may be horizontal before the load is applied but may not be perfectly fixed. In that case the total deflection includes the deflection of the tangent line from its position before the load was applied. The deflection of the tangent line BC (Fig. 113) from the horizontal line at any distance x_1 from C is the product of the slope of the tangent multiplied by x_1. If the slope is positive, as in the figure, the deflection is positive when x_1 is measured toward the right, and negative when it is measured toward the left. For small angles, such as occur in beams, the distance measured along the slope is taken as equivalent to its horizontal projection. BC is taken as equal to AC.

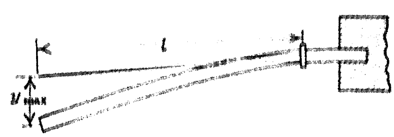


FIG. 114.—Cantilever deflection.

It is difficult to fix a beam so that the slope of the tangent at the wall will not change when the load is applied. For this reason cantilevers are not generally used to determine modulus of elasticity. Accurate results may be obtained by clamping a pointer to the beam at some fixed point (as in Fig. 114) and

measuring the deflection of the cantilever with respect to this pointer.

The equations of Articles 76 and 77 apply to a portion of a cantilever extending from the free end to any section and give the deflection of the portion of the beam from the tangent at the section.

A beam may be loaded at some distance from the free end. The part between the free end and the load remains straight, and its deflection is calculated by multiplying its length by the slope at the load.

Example

A beam of length $a + b$ has a load P at a distance a from the free end. Find the deflection at the free end. From Formula XVI the deflection under

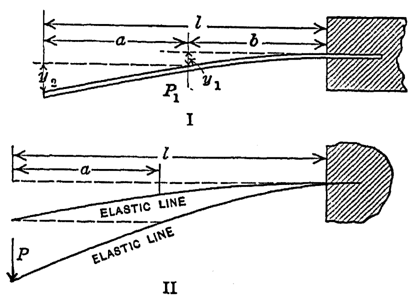


FIG. 115.—Cantilever with concentrated load.

the load is $-\dfrac{Pb^3}{3\,EI}\cdot$ The deflection (y_2 of Fig. 115), due to the straight portion of length a is $-a\dfrac{dy}{dx}$ where $\dfrac{dy}{dx}$ is the slope under the load. From equation (4) of Article 76,

$$\frac{dy}{dx} = \frac{Pb^2}{2\,EI}, \quad -a\,\frac{dy}{dx} = -\frac{Pab^2}{2\,EI}.$$

$$\text{The total deflection} = -\frac{Pb^2}{6\,EI}(3\,a + 2\,b)$$

$$= \frac{Pb^2}{6\,EI}(2l + a) = \frac{P(l - a)^2}{6\,EI}(2\,l + a) \tag{1}$$

This total deflection is the same as the answer of Problem 7 of Article 76. The deflection at the end of a cantilever due to a load at a distance a from the free end is equal to the deflection at a distance a from the free end due to the same load on the end, Fig. 115 II.

Problem

A cantilever of length $a + b$ carries a distributed load of w per unit length for a length b from the fixed end and no load on the remainder. Find the deflection at the free end. *Ans.* Deflection $= \frac{wb^3}{24EI}(4a + 3b)$.

79. Beam Supported at the Ends, Uniformly Loaded.—In a beam supported at the ends and uniformly loaded, the end reactions are each equal to one-half of the total load,

$$R_1 = R_2 = \frac{W}{2} = \frac{wl}{2}.$$

The moment at a distance x from the left support is

$$\frac{wlx}{2} - \frac{wx^2}{2},$$

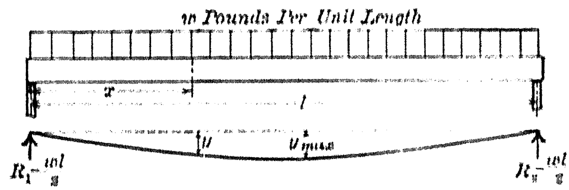


FIG. 116.—Supports at ends, load uniformly distributed.

and the differential equation becomes:

$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2}. \tag{1}$$

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + C_1. \tag{2}$$

From symmetry it is evident that the maximum deflection is at the middle

$$\frac{dy}{dx} = 0 \quad \text{when} \quad x = \frac{l}{2};$$

$$C_1 = -\frac{wl^3}{24}. \tag{3}$$

$\left(\frac{C_1}{EI}\right.$ is the slope of the elastic line at the left support.$\left.\right)$

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24}. \tag{4}$$

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} + C_2. \tag{5}$$

At $x = 0$, $y = 0$; $C_2 = 0$; (6)

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}. \tag{7}$$

When $x = \frac{l}{2}$ the deflection is a maximum,

$$EIy_{\max} = \frac{wl^4}{12}\left(\frac{1}{8} - \frac{1}{32} - \frac{1}{4}\right) = -\frac{5wl^4}{384};$$

$$y_{\max} = -\frac{5\,wl^4}{384\,EI} = -\frac{5\,Wl^3}{384\,EI}. \qquad \text{Formula XVIII.}$$

Substituting $x = l$ in equation (7), the deflection at the right support is found to be zero. This condition might have been used to determine one of the constants.

This beam might be regarded as fixed at the middle where the slope is zero, and to consist of two cantilevers of length $\frac{l}{2}$ which are bent downward by the distributed load and bent upward by the end reactions. The deflection at one end is:

$$\text{Downward,} \quad \frac{\frac{W}{2}\left(\frac{l}{2}\right)^3}{8\,EI} = \frac{Wl^3}{128\,EI},$$

$$\text{Upward,} \quad \frac{\frac{W}{2}\left(\frac{l}{2}\right)^3}{3\,EI} = \frac{Wl^3}{48\,EI};$$

$$\text{Total deflection upward,} \quad \frac{Wl^3}{384\,EI}(8 - 3) = \frac{5\,Wl^3}{384\,EI}.$$

The deflection at any point, measured upward from the middle, may be calculated in a similar way.

Problems

1. What is the deflection at the middle of a 2-inch by 10-inch floor joist, 12 feet between supports, due to a load of 90 pounds per foot, if E is 1,200,000 pounds per square inch? *Ans.* 0.210 inch.

2. In Problem 1 what is the slope of the tangent to the beam at the supports? If the beam extends 5 feet beyond the support, how much is the end elevated when the load is applied between the supports?

Ans. 0.00467; 0.28 inch.

3. A 20-inch 65-pound I-beam is used with a span of 25 feet to carry a load of 1,200 pounds per foot. If E is 29,000,000 pounds per square inch what is the deflection? What is the maximum fiber stress?

Ans. Deflection at middle, 0.311 inch.

4. What is the greatest deflection in a beam of length l, depth d, and

modulus of elasticity E for a beam supported at the ends with a uniformly distributed load, if the unit stress shall not exceed an allowable value of S?

Ans. Deflection $= \dfrac{5Sl^2}{24\,Ed}.$

✓**80. Beam Supported at the Ends with Load at the Middle.**—If P is the load at the middle, the reactions are $\frac{P}{2}$, and the moment from the left end to the middle is $\frac{Px}{2}$.

$$EI\frac{d^2y}{dx^2} = \frac{Px}{2}. \tag{1}$$

$$EI\frac{dy}{dx} = \frac{Px^2}{4} + C_1. \tag{2}$$

At the middle, from the symmetry of the sides, $\frac{dy}{dx} = 0$;

$$C_1 = -\frac{Pl^2}{16}. \tag{3}$$

$$EI\frac{dy}{dx} = \frac{Px^2}{4} - \frac{Pl^2}{16}. \tag{4}$$

$$EIy = \frac{Px^3}{12} - \frac{Pl^2x}{16} + C_2. \tag{5}$$

At the left support, where $x = 0$, $y = 0$;

$$C_2 = 0. \tag{6}$$

$$EIy = \frac{Px^3}{12} - \frac{Pl^2x}{16}. \tag{7}$$

Equation (7) holds good from the left end to the middle. Beyond the middle the moment equation is changed. At the middle where $x = \frac{l}{2}$,

$$y_{\max} = \frac{Pl^3}{96\,EI} - \frac{Pl^3}{32\,EI} = -\frac{Pl^3}{48\,EI}. \quad \text{Formula XIX.}$$ ✓

This beam may be regarded as fixed at the middle and to consist of two cantilevers of length $\frac{l}{2}$ bent upward by the reaction $\frac{P}{2}$. If this length and reaction are substituted for l and P of Formula XVI the result is Formula XIX.

Formula XIX is much used to determine the modulus of elasticity.

Problems

1. A selected beam of red oak, 1.75 inches wide and 1.25 inches deep, was placed on two supports 12 inches apart and a load applied at the middle. When an addition of 607 pounds was made to the load, the deflection at the middle was increased 0.050 inch. Find E.

Ans. 1,534,000 pounds per square inch.

2. In the beam of Problem 1 an addition of 721 pounds produced a deflection of 0.059 inch. Find E.

3. In Problem 1 how much would the last significant figures of the value of E be changed if the deflection readings were incorrect 0.0005 inch? if the breadth were incorrect 0.005 inch? if the depth were incorrect 0.005 inch? if the load were incorrect 1 pound? How much would E be changed if all these errors occurred at once in the same direction?

Ans. An error of 0.005 inch in the depth would change the result 1.2 per cent., a change of 18 in the significant figures.

4. The beam of Problem 1 broke under a load of 2,315 pounds. Find the fiber stress at rupture.

5. A 12-inch 31.5-pound I-beam rests on two supports 15 feet apart and overhangs one support 5 feet. Find the deflection at the middle of the span and the elevation of the overhanging end when a load of 8,000 pounds is applied at the middle of the span. E is 29,000,000 pounds per square inch.

Ans. 0.155 inch.

6. Find the deflection at the middle of an 18-inch 55-pound I-beam, supported at the ends, for a span of 20 feet, due to a distributed load of 1,200 pounds per foot and a load of 10,000 pounds at the middle if E is 29,000,000 pounds per square inch. *Ans.* 0.312 inch.

7. Substitute $x = l$ in equation (7). The result is not the deflection at the right support. Why?

81. Beam Supported at Ends, Load at Any Point between Supports.—The moment equation changes at each concentrated load. The differential equation (1) and the equation of the elastic line (7) of the preceding article applies only for half the beam, from the left support to the middle. Fortunately, on account of the symmetry of the two portions, it may be assumed that the beam is horizontal under the load, thus securing the two conditions necessary for the determination of the arbitrary constants. When the load is not at the middle, the beam is no longer symmetrical, and the location of the point where $\frac{dy}{dx}$ is zero is not apparent. In that case the differential equations must be written for both portions of the beam, and four arbitrary constants determined.

Fig. 117 represents a beam of length l supported at the ends

with a load P at a distance a from the left end. If $l - a = b$, the left reaction is $\frac{Pb}{l}$. For the left portion of the beam

$$M = \frac{Pbx}{l},$$

and to the right of the load

$$M = \frac{Pbx}{l} - P(x - a).$$

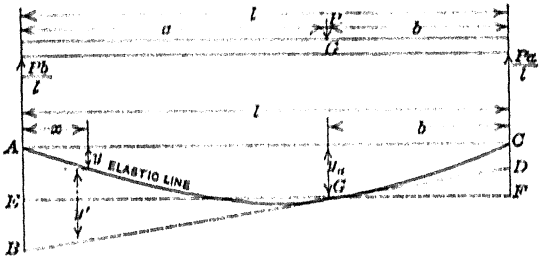


FIG. 117.—Beam with load at any point.

For all points from $x = 0$ to $x = a$, inclusive,

$$EI\frac{d^2y}{dx^2} = \frac{Pbx}{l}. \qquad (1)$$

$$EI\frac{dy}{dx} = \frac{Pbx^2}{2\,l} + C_1. \qquad (3)$$

For all points from $x = a$ to $x = l$, inclusive,

$$EI\frac{d^2y}{dx^2} = \frac{Pbx}{l} - P(x - a). \qquad (2)$$

$$EI\frac{dy}{dx} = \frac{Pbx^2}{2\,l} - \frac{P(x - a)^2}{2} + C_3. \qquad (4)$$

The curve is continuous under the load with no abrupt change of slope. When $x = a$ the value of $\frac{dy}{dx}$ calculated from (3) is the same as when calculated from (4). This makes the first members of the two equations equal and, consequently, the second members are equal when a is substituted for x.

$$\frac{Pba^2}{2\,l} + C_1 = \frac{Pba^2}{2\,l} - \frac{P(a - a)^2}{2} + C_3;$$

$$C_1 = C_3.$$

Substituting C_1 for C_3 in equation (4) and integrating both equations,

$$EIy = \frac{Pbx^3}{6\,l} + C_1x + C_2. \qquad (5)$$

When $x = 0$, $y = 0$, hence $C_2 = 0$.

$$EIy = \frac{Pbx^3}{6\,l} - \frac{P(x - a)^3}{6} + C_1x + C_4. \qquad (6)$$

When $x = a$ the values of y from (5) and (6) are the same and the second members of these equations are equal, from which:

$$0 = C_2 = C_4.$$

When $x = l$ in (6), $y = 0$;

$$C_1 = -\frac{Pbl^2}{6\,l} + \frac{P(l-a)^3}{6\,l} = -\frac{Pb}{6\,l}(l^2 - b^2). \tag{7}$$

Substituting the value of C_1 from (7) in (5),

$$EIy = \frac{Pbx^3}{6\,l} - \frac{Pb(l^2 - b^2)x}{6\,l}. \tag{8}$$

Substituting C_1 in (3) and equating to zero,

$$x^2 = \frac{l^2 - b^2}{3} = \frac{a(a + 2\,b)}{3}, \tag{9}$$

gives the point of maximum deflection, provided b is less than a. Substituting x from (9) in (5),

$$y_{\max} = -\frac{Pb\,(l^2 - b^2)\sqrt{3\,(l^2 - b^2)}}{27\,EIl} = -\frac{Pba(a + 2\,b)\sqrt{3a(a + 2\,b)}}{27\,EIl}. \tag{10}$$

The deflection under the load is

$$y = -\frac{Pa^2b^2}{3\,EIl}. \tag{11}$$

Problems

1. Show that the point of maximum deflection is never beyond $\dfrac{l}{\sqrt{3}}$.

2. A 3-inch by 2-inch rectangular beam, 10 feet long, supports a load of 45 pounds 6 feet from the left end. Find the deflection at the middle, under the load, and at the point of maximum deflection if the beam is supported at the ends and E is 1,500,000 pounds per square inch.
Ans. At middle, 0.510 inch; under load, 0.498 inch; maximum, 0.512 inch.

82. Beam Supported at the Ends, Two Equal Loads Symmetrically Placed.—An important case is that of a beam supported at the ends with two equal loads at equal distances from the supports. If the weight of the beam is neglected, the shear is zero and the moment is constant between the supports. For

this reason it is much used in tests of beams, for it enables the experimenter to study the effect of moment without shear between the loads and the combined effect of moment and shear between the loads and the supports. With constant moment between the supports, the unit stress in any horizontal fiber is constant, and measurements of elongation may be used to locate the neutral surface.

In Fig. 118 each load is $\frac{P}{2}$, the length of the span is l, and the loads are at a distance a from the left and right supports, respectively. There are three moment equations, but on account of the symmetry it is evident that $\frac{dy}{dx}$ is zero at the middle so that two

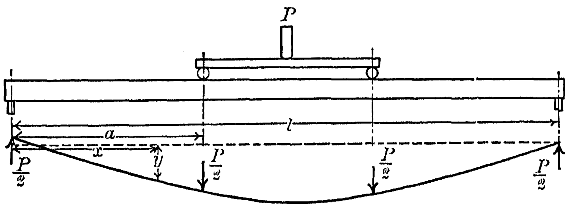


FIG. 118.—Supports at ends, two loads symmetrically placed.

equations will be sufficient. Four conditions are needed to determine the four constants of these two equations. These are $y = 0$ when $x = 0$ in the first portion; $\frac{dy}{dx}$ has the same value for both equations under the first load; y has the same value for both equations under this load; and $\frac{dy}{dx} = 0$ at the middle.

Writing the equations as in the preceding article:

From $x = 0$ to $x = a$,		From $x = a$ to $x = (l - a)$,	
$EI\frac{d^2y}{dx^2} = \frac{Px}{2}.$	(1)	$EI\frac{d^2y}{dx^2} = \frac{Pa}{2}.$	(2)
$EI\frac{dy}{dx} = \frac{Px^2}{4} + C_1.$	(3)	$EI\frac{dy}{dx} = \frac{Pax}{2} + C_3.$	(4)

$$\frac{Pa^2}{4} + C_1 = \frac{Pa^2}{2} + C_3;$$

$$C_1 = \frac{Pa^2}{4} + C_3.$$

When $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$;

$$C_3 = -\frac{Pal}{4}.$$

$$C_1 = \frac{Pa^2}{4} - \frac{Pal}{4}.$$

$$EI\frac{dy}{dx} = \frac{Px^2}{4} + \frac{Pa^2}{4} - \frac{Pal}{4}.$$

$$EIy = \frac{Px^3}{12} + \frac{Pa^2x}{4} - \frac{Palx}{4} + C_2. \quad (5)$$

$$C_2 = 0.$$

$$EI\frac{dy}{dx} = \frac{Pax}{2} - \frac{Pal}{4}.$$

$$EIy = \frac{Pax^2}{4} - \frac{Palx}{4} + C_4. \quad (6)$$

$$\frac{Pa^3}{12} + \frac{Pa^3}{4} - \frac{Pa^2l}{4} = \frac{Pa^3}{4} - \frac{Pa^2l}{4} + C_4;$$

$$C_4 = \frac{Pa^3}{12}$$

$$EIy = \frac{Px^3}{12} + \frac{Pa^2x}{4} - \frac{Palx}{4}. \quad (7)$$

$$EIy = \frac{Pax^2}{4} - \frac{Palx}{4} + \frac{Pa^3}{12}. \quad (8)$$

At the middle, where $x = \frac{l}{2}$,

$$y_{\max} = \frac{Pa}{EI}\left(\frac{a^2}{12} - \frac{l^2}{16}\right). \quad (9)$$

With the loads at the third points so that $a = \frac{l}{3}$,

$$y_{\max} = \frac{Pl^3}{3\,EI}\left(\frac{1}{108} - \frac{1}{16}\right) = -\frac{23\,Pl^3}{1{,}296\,EI}.$$

At the fourth points where $a = \frac{l}{4}$,

$$y_{\max} = -\frac{11\,Pl^3}{768\,EI}.$$

Problems

1. Verify equation (9) by substituting $a = \frac{l}{2}$.

2. A 6-inch by 8-inch wooden beam supported at points 12 feet apart is

loaded with two equal loads of 800 pounds each 4 feet from the supports. If E is 1,200,000, what is the deflection under a load and at the middle?

Ans. Under a load, 0.240 inch; at middle, 0.276 inch.

3. In Problem 2, two vertical lines are ruled on one side of the beam 20 inches on the right and left of the middle. When the load is applied, what angles will these lines make with each other?

4. In Problem 3, Fig. 119, the distance FN between the upper ends of the lines is measured with a delicate extensometer. How much is this distance diminished when the loads are applied, and what is the unit deformation in 40 inches? *Ans.* Total, 0.0200 inch; unit, 0.00050 inch.

5. Compute the fiber stress in the upper fibers from the unit deformation in Problem 4 and check by Formula XIII.

6. Show that the error due to measuring the chord instead of the arc in Problem 4 is less than 0.00004 inch, and that the relative error in the unit deformation is less than 1 part in 500. (Use the first two terms of the sine series for the half-angle.)

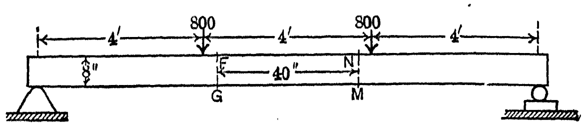


FIG. 119.—Beam loaded at third points.

83. Any Beam with Two Supports.—All the cases of deflections so far considered are cases of two supports. (In the cantilever one of these supports pushes *down* in the wall.) In all problems of this sort, the reactions may be computed algebraically and the moment equations written for any section. At each support and at each concentrated load the equation of moment changes and a different differential equation must be formed. The solution of each differential equation of the second order involves two integration constants which must be determined from the values of y and $\frac{dy}{dx}$ at points to which the equations apply. There must be twice as many of these known conditions as there are differential equations.

Fig. 120 represents a beam overhanging two supports. If the load between the supports is uniformly distributed, the equation of the elastic line for that portion may be obtained from the solution of one differential equation, since $y = 0$ at the two supports furnishes the two conditions necessary to determine the integration constants. If there is a single concentrated load between the supports, there are two differential equations to be solved and the conditions are those of Article 81. The equations for the

overhanging parts cannot be determined without taking into account the portion between the supports. One condition for each of these is $y = 0$ over the support; the second condition is the slope over the support, which is obtained from the equation of the adjacent part between the supports. The part to the left of the left support may be treated as a cantilever and its deflection from the tangent line AB may be calculated by the formulas

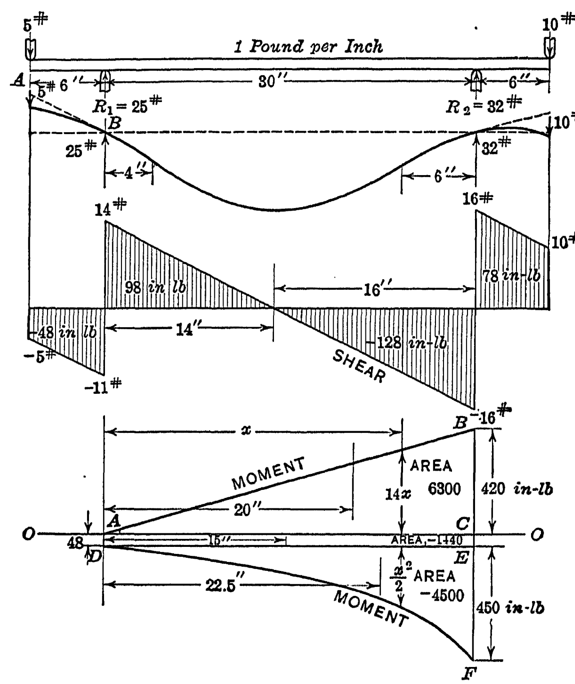


FIG. 120.—Beam overhanging supports.

of Article 76. The deflection of this tangent line is determined from its slope and the distance from B.

Fig. 120 is a beam 42 inches long, weighing 1 pound per inch, supported 6 inches from the ends, with a load of 5 pounds on the left end and a load of 10 pounds on the right end. The left reaction is 25 pounds and the right reaction is 32 pounds. The shear is zero and the moment a maximum at 14 inches from the left support. For the part between the supports, using the general moment equation,

$$EI\frac{d^2y}{dx^2} = -48 + 14x - \frac{x^2}{2}. \tag{1}$$

$$EI\frac{dy}{dx} = -48x + 7x^2 - \frac{x^3}{6} + C_1. \tag{2}$$

$$EIy = -24x^2 + \frac{7x^3}{3} - \frac{x^4}{24} + C_1x + [C_2 = 0]. \tag{3}$$

$$C_1 = 24 \times 30 - \frac{7 \times 30^2}{3} + \frac{30^3}{24} = -255,$$

$C_1 = -255$ = slope of EIy curve at left support.

Substituting in (2), $EI\frac{dy}{dx} = 105$ at the right support.

For the part to the left of the left support the deflection from the tangent AB is calculated by the formulas for a cantilever with uniformly distributed load and a cantilever with a concentrated load on the free end. The total deflection is the sum of these deflections downward and the upward deflection of AB

Problems

1. In Fig. 120 find the deflection at the left end.

Ans. $EIy = 1{,}530 - 360 - 162 = 1{,}008.$

2. In Fig. 120 find the deflection at the right end.

Ans. $EIy = 630 - 720 - 162 = -252.$

DEFLECTION OF BEAM WITH TWO SUPPORTS BY METHOD OF AREA MOMENTS

(*The remainder of this chapter may be omitted if Article 73 to 83 have been studied.*)

84. Principles of Method of Area Moments.—The calculation of the deflection of beams by the method of *area moments** has decided advantages in some cases, and is used by an increasing number of engineers. While apparently very different from the method of double integration, the real variation lies in the order in which the operations are performed, and in the use of limits instead of integration constants. Formula XV may be written

$$y = \int\int\frac{d^2y}{dx^2}dx\,dx = \int\frac{M}{EI}\left(\int_{x_1}^{x}dx\right)dx = \int\frac{M}{EI}\Big[x\Big]_{x_1}^{x} \tag{1}$$

$$y = \int\frac{M}{EI}(x - x_1)\,dx. \qquad \text{Formula XX.}$$

*This method was devised by Mohr and independently in America by Prof. Charles E. Greene, who began to teach it in 1873. See paper by A. E. GREENE in the *Michigan Technic* of June, 1910.

The dx inside the parenthesis is integrated first, and then, after substituting the limits (one of which is the constant x_1 and the other is the variable x) this is combined with the remaining terms for the second integration.

If x_1 is zero, Formula XX becomes,

$$y = \int \frac{M}{EI} x \, dx. \qquad \text{Formula XXI.}$$

The method is sometimes called the integral of $\frac{M}{EI}x\,dx.$ method, or, when I is constant, the integral of $Mx\,dx$ method. Starting with Formulas XX or XXI it involves only one integration.

85. Deflection in Terms of the Moment Diagram.—If the diagram for $\frac{M}{EI}$ be plotted as in Fig. 121, an element of its area is $\frac{M}{EI}\,dx$ and the product $\frac{M}{EI}(x - x_1)dx$ is the moment of this element

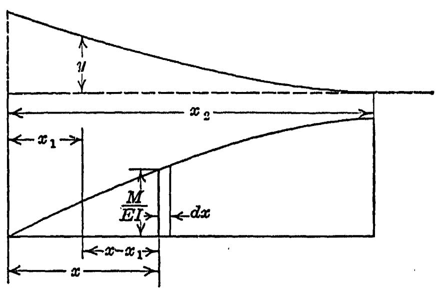


FIG. 121.—Area moments.

of area with respect to the point whose abscissa is x_1. The total moment with respect to x_1 of that part of the diagram between x_1 and x_2 is given by

$$\text{Total moment} = \int_{x_1}^{x_2} \frac{M}{EI}(x - x_1)dx. \qquad (1)$$

The second member of equation (1) is also the second member of Formula XX, consequently the first members are equal and

$$y = \text{total moment} = \int_{x_1}^{x_2} \frac{M}{EI}(x - x_1)dx. \qquad (2)$$

The deflection of a point x_1 is equal to the moment with respect to x_1 of that part of the $\frac{M}{EI}$ diagram between x_1 and x_2.

If the moment of inertia is constant, the moment diagram may be used instead of the $\frac{M}{EI}$ diagram, so that for beams of constant section,

$$EIy = \int_{x_1}^{x_2} M(x - x_1)dx, \tag{3}$$

and the moment with respect to x_1 of that portion of the moment diagram between x_1 and x_2 is equal to the deflection of the point x_1 from the tangent at x_2, multiplied by EI.

If x_1 is 0, equation (3) becomes

$$EIy = \int_0^{x_2} Mx\,dx, \tag{4}$$

when the section is uniform, and equation (2) for any beam of constant section becomes

$$y = \frac{1}{EI}\int_0^{x_2} Mx\,dx. \tag{5}$$

86. Geometrical Meaning of the Deflection Integral.—In Fig. 122 AB represents a portion of a beam. A_1B_1B represents the

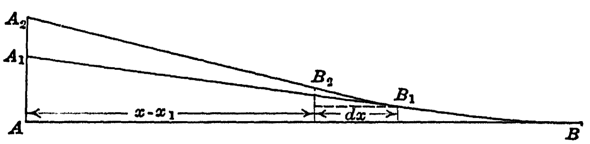


FIG. 122.—Increment of deflection.

same portion with that part between B_1 and B bent by a moment M, the part A_1B_1 remaining straight and tangent to B_1B. If the part between B_1 and B_2 is now bent, the point A_1 moves to A_2. (It is assumed that the deflection is small so that A, A_1 and A_2 are on the same straight line perpendicular to AB.) Let x_1 be the abscissa of A and A_1, and let x be the abscissa of B_1, so that $x - x_1$ is the horizontal distance from A to B_1. When the part between B_1 and B_2 is bent, the additional deflection A_1A_2 is $x - x_1$ multiplied by the change in slope of the tangents at B_1 and B_2. If the horizontal distance from B_2 to B_1 is dx, the change in slope is $\frac{d^2y}{dx^2}\,dx$, so that the increment of deflection A_1A_2 is given by

$$dy = (x - x_1)\frac{d^2y}{dx^2}dx = \frac{M}{EI}(x - x_1)dx, \tag{1}$$

since

$$\frac{d^2y}{dx^2} = \frac{M}{EI},$$

and

$$y = \int \frac{M}{EI}(x - x_1)dx. \qquad \text{Formula XX.}$$

If Formula XX be integrated from $x = x_1$ to $x = x_2$, the result is the *deflection of the point x_1 from a line tangent to the beam at x_2.*

The application of the *area moments method* in the examples of Articles 87 to 93, is subdivided into four parts numbered as follows:

I. *The deflection at one point by means of the geometry of the moment diagram.*

II. *The deflection at one point by integration.*

III. *The general expression for deflection by means of the geometry of the moment diagram.*

IV. *The general expression for deflection by integration.*

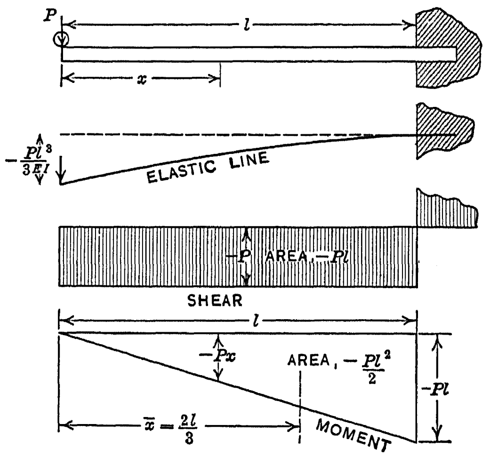


FIG. 123.—Area moments for cantilever.

87. Cantilever with Load on the Free End.—Fig. 123 represents a cantilever of uniform section with a load P on the free end. The moment at a distance x from the free end taken as the origin is $-Px$. The moment diagram is a triangle, the area of which is regarded as negative.

I. The entire moment is a triangle of base l, altitude $-Pl$,

and area $-\frac{Pl^2}{2}$. The horizontal distance of the center of gravity of this moment triangle from the left end of the beam is $\frac{2l}{3}$. The deflection of the left end below the tangent at the right end is given by

$$EIy_{\max} = -\frac{Pl^2}{2} \times \frac{2l}{3} = -\frac{Pl^3}{3}; \qquad (1)$$

$$y_{\max} = -\frac{Pl^3}{3EI}. \qquad \text{Formula XVI.}$$

II. To find the deflection at the free end by integrating Formula XXI, $M = -Px$, $Mx = -Px^2$.

$$\int Mx\,dx = -P\int x^2dx = -\frac{P}{3}\Big[x^3\Big]_0^l = -\frac{Pl^3}{3}. \qquad (2)$$

$$EIy_{\max} = -\frac{Pl^3}{3}, \quad y_{\max} = -\frac{Pl^3}{3EI}.$$

Problems

1. Find the deflection at the end of an 8-inch 18-pound I-beam as a cantilever 10 feet long, due to a load of 1,200 pounds on the end, if E is 29,000,000 pounds per square inch. What is the maximum fiber stress at the dangerous section due to this load and the weight of the beam?

Ans. Deflection at end, 0.419 inch; fiber stress, 10,900 lb./in.^{2}

2. A wooden cantilever 6 inches square and 10 feet long is deflected 0.6 inch at the end by a load of 135 pounds at the end. Find E.

Ans. E, 1,200,000 pounds per square inch.

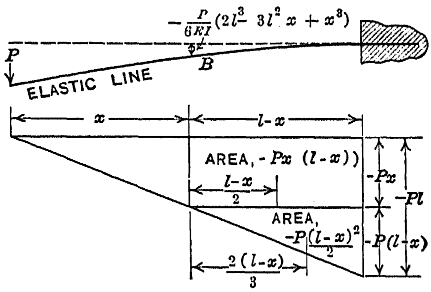


FIG. 124.—Area moments for any point of cantilever.

III. Fig. 124 is the moment diagram used to find the deflection at a distance x from the free end. The area to the right of the point B is made up of the rectangle of base $l - x$ and altitude

$-Px$, and the lower triangle of the base $l - x$ and altitude $-P(l - x)$. The moment with respect to B of these areas:

$$\text{Rectangle, } -Px(l - x) \times \frac{l - x}{2} = -\frac{Px}{2}(l - x)^2. \quad (3)$$

$$\text{Triangle, } -\frac{P(l - x)^2}{2} \times \frac{2(l - x)}{3} = -\frac{P(l - x)}{3}(l - x)^2. \quad (4)$$

$$\text{Total moment} = -\frac{Pl^3}{3} + \frac{Pl^2x}{2} - \frac{Px^3}{6}. \quad (5)$$

$$y = -\frac{P}{6EI}(2l^3 - 3l^2x + x^3). \quad (6)$$

IV. To find the deflection at any point by integration, the abscissa of the point will be called x_1 and Formula XX applied.

$$\int M(x - x_1)\,dx = -P\int x(x - x_1)\,dx = -P\int x^2dx + P\int x_1x dx;$$

$$EIy = -\left[\frac{Px^3}{3} - \frac{Px_1x^2}{2}\right]_{x_1}^{l} = -P\left(\frac{l^3}{3} - \frac{l^2x_1}{2} + \frac{x_1^3}{6}\right). \quad (7)$$

Dropping the subscript, equation (7) becomes equation (6)

Problem

3. If the deflection at the end of a cantilever due to a load on the end is 1 inch what is the deflection at 0.4 the length from the fixed end?

88. Deflection at the End of a Cantilever Due to a Load at Any Point.—I. Let a, Fig. 125, be the distance of the load from the free end. The remainder of the length is $l - a$; the moment triangle has an area $-P\dfrac{(l - a)^2}{2}$; and the moment arm with respect to the free end is

$$a + \frac{2(l - a)}{3}, \text{ which reduces to } \frac{2l + a}{3}.$$

$$EIy = -P\frac{(l - a)^2}{2} \times \frac{2l + a}{3} = -\frac{P}{6}(2l^3 - 3l^2a + a^3). \quad (1)$$

$$y_{\max} = -\frac{P}{6EI}(2l^3 - 3l^2a + a^3). \quad (2)$$

Equation (2) is identical with equation (6) of the preceding article, with distance from the end represented by a instead of by x. The deflection at *any* point A of a cantilever due to a

load at B is equal to the deflection at B due to the same load at A.

II. Taking the origin at the load, $M = -Px$, $x_1 = -a$,

$$\int M(x - x_1)dx = -P\int (x^2 + ax)dx = -P\left[\frac{x^3}{3} + \frac{ax^2}{2}\right]_0^{l-a} \quad (1)$$

The limits are $x = 0$ and $x = l - a$ as the bending takes place only in the part subjected to bending moment.

$$EIy = -\frac{P}{6}\left[2\,(l - a)^3 + 3\,a\,(l - a)^2\right] \quad (2)$$

$$y_{max} = -\frac{P}{6}\left(2\,l + a\right)\left(l - a\right)^2 = -\frac{P}{6\,EI}\,(2\,l^3 - 3\,l^2a + a^3). \quad (3)$$

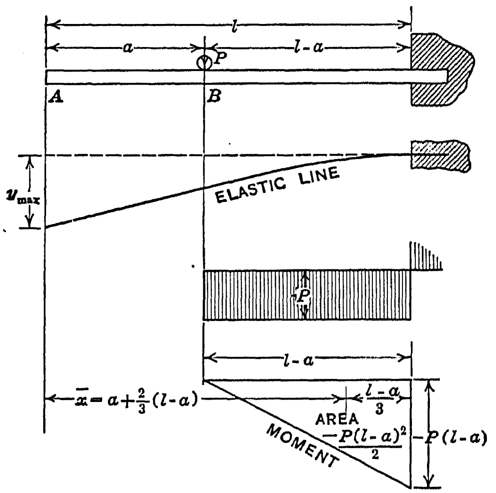


FIG. 125.—Cantilever with load at any point.

Problems

1. A 4-inch by 4-inch wooden cantilever 15 feet long has a load of 40 pounds 5 feet from the free end and a load of 60 pounds 10 feet from the free end. Find the deflection at the free end due to these loads if E is 1,000,000 pounds per square inch. *Ans.* 2.7 inches.

2. Derive equation (3) by integration, taking the free end of the beam as the origin.

89. Cantilever with Uniformly Distributed Load.—The moment at a distance x from the free end is $-\frac{wx^2}{2}$, where w is the load per unit length.

I. The entire moment diagram is the parabola, Fig. 126, the maximum ordinate of which is $-\frac{wl^2}{2}$. The area is $-\frac{wl^3}{6}$, being the same as the volume of a pyramid of base $\frac{wl^2}{2}$ and altitude l. For the center of gravity $\bar{x} = \frac{3\,l}{4}$, which is the same as the distance of the center of gravity of a pyramid from the vertex.

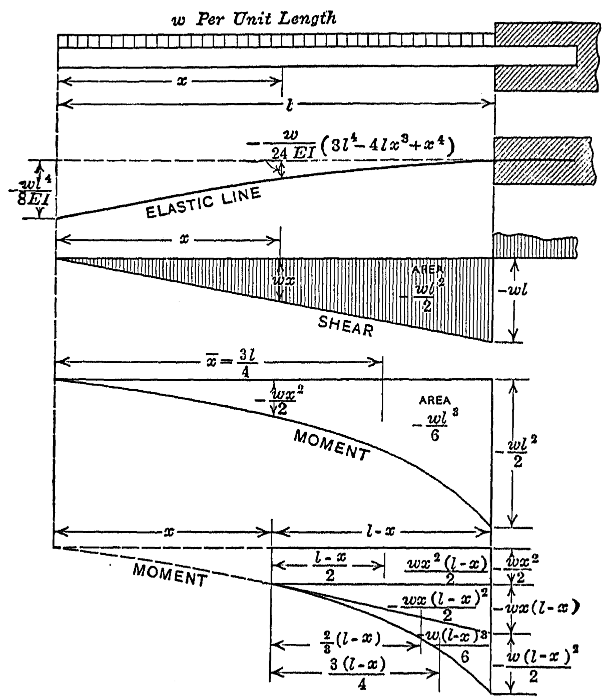


FIG. 126.—Cantilever with distributed load.

$$EIy_{\max} = -\frac{wl^3}{6} \times \frac{3\,l}{4} = -\frac{wl^4}{8} \tag{1}$$

$$y_{\max} = -\frac{wl^4}{8\,EI} = -\frac{Wl^3}{8\,EI}, \quad \text{Formula XVII.}$$

where $W = wl$ is the total distributed load.

II. $$\int Mxdx = -w\int\frac{x^3}{2} = -w\left[\frac{x^4}{8}\right]_0^l = -w\frac{l^4}{8} = EIy_{\max}. \tag{2}$$

$$y_{\max} = -\frac{wl^4}{8\,El} = -\frac{Wl^3}{8\,EI}.$$

III. The lower diagram of Fig. 126 represents the moment between the fixed end and a point at a distance x from the free end. This diagram may be subdivided into three parts, each of which has the horizontal length of $l - x$. These are, the rectangle of height $\frac{wx^2}{2}$, the triangle of altitude $wx(l - x)$, and the lower figure of altitude $\frac{w(l-x)^2}{2}$, which is equivalent to a parabola.

Area	Moment arm	Moment
$-\frac{wx^2(l-x)}{2}$	$\frac{l-x}{2}$	$-\frac{wx^2(l-x)^2}{4}$
$-\frac{wx(l-x)^2}{2}$	$\frac{2(l-x)}{3}$	$-\frac{wx(l-x)^3}{3}$
$-\frac{w(l-x)^3}{6}$	$\frac{3(l-x)}{4}$	$-\frac{w(l-x)^4}{8}$

$$EIy = -\frac{w}{24}(3\,l^2 + 2\,lx + x^2)\,(l - x)^2. \tag{3}$$

$$y = -\frac{w}{24\,EI}(3\,l^4 - 4\,l^3x + x^4). \tag{4}$$

IV. With the origin at the free end,

$$M = -\frac{wx^2}{2},\ M(x - x_1) = \frac{wx^2}{2}(x - x_1).$$

$$EIy = -\frac{w}{2}\int(x^3 - x_1x^2)\,dx = -\frac{w}{2}\left[\frac{x^4}{4} - \frac{x_1x^3}{3}\right]_{x_1}^{l} \tag{5}$$

$$EIy = -\frac{w}{2}\left(\frac{l^4}{4} - \frac{x_1l^3}{3} - \frac{x_1^4}{4} + \frac{x_1^4}{3}\right). \tag{6}$$

Dropping the subscripts,

$$y = -\frac{w}{24\,EI}\left(3\,l^4 - 4\,l^3x + x^4\right). \tag{4}$$

Problem

How does the deflection at the end of a cantilever due to a load uniformly distributed, compare with the deflection due to a load on the end, if the maximum fiber stress is the same in both cases?

90. Cantilever with Distributed Load on Part Adjacent to Fixed End.—I. In Fig. 127 the portion of length a from the free end is not loaded. The area of the moment diagram is $-\frac{w(l-a)^3}{6}$

and the moment arm of the area with respect to the end of the beam is

$$a + \frac{3}{4}(l - a) = \frac{3\,l + a}{4}. \tag{1}$$

$$EIy_{\max} = -\frac{w(l - a)^3}{6} \times \frac{3\,l + a}{4}, \tag{2}$$

$$y_{\max} = -\frac{w}{24\,EI}(3\,l^4 - 8\,l^3a + 6\,l^2a^2 - a^4). \tag{3}$$

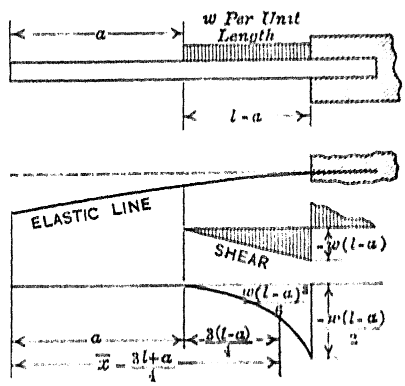


FIG. 127.—Cantilever with distributed load over part of length.

Problems

1. If $a = \frac{l}{2}$, show that the deflection at the free end is

$$y_{\max} = -\frac{7\,wl^4}{384\,EI}.$$

2. Derive the formula for the deflection if the length a from the free end has a load of w per unit length and the remainder is not loaded.

91. Beam Supported at the Ends with Load at the Middle.—From symmetry it is evident that the beam is horizontal at the middle.

I. The area of the triangle EFG, Fig. 128, which represents the moment from the left end to the middle, is $\frac{Pl^2}{16}$, and its center of gravity is $\frac{l}{3}$ from the left end. The deflection of the left end upward from the tangent at the middle is given by

$$EIy_{\max} = \frac{Pl^2}{16} \times \frac{l}{3} = \frac{Pl^3}{48}, \tag{1}$$

$$y_{\max} = \frac{Pl^3}{48\,EI}. \qquad \text{Formula XIX.}$$

The slope of the tangent at the right end may be calculated by means of the whole moment diagram. To find AC, which is the deflection at the left end from the line CB which is tangent to the beam at the right end, the area of the two triangles is $\frac{Pl^2}{8}$ and the moment arm is $\frac{l}{2}$, so that $EIy = \frac{Pl^3}{16}$. Dividing by l, the slope is found to be $\frac{Pl^2}{16\,EI}$.

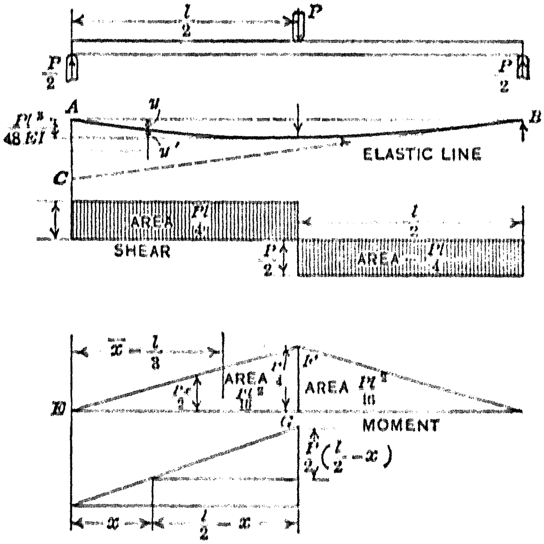


FIG. 128.—Beam supported at ends, load at middle.

III. The moment diagram is made up of the rectangle of altitude $\frac{Px}{2}$ and the triangle of altitude $\frac{P}{2}\left(\frac{l}{2} - x\right)$. The base of each is $\frac{l}{2} - x$. Multiplying the area of the rectangle by one-half its base and the area of the triangle by two-thirds its base, and adding,

$$EIy' = \frac{P}{12}(l + x)\left(\frac{l}{2} - x\right)^2 = \frac{P}{48}(l^3 - 3\,l^2x + 4\,x^3). \qquad (2)$$

where y' is the deflection upward from the tangent at the middle.

To find y, the deflection downward from the position before the load was applied, Formula XIX is added to y' from equation (2),

$$y = -\frac{Pl^3}{48\,EI} + \frac{Pl^3}{48\,EI} - \frac{Pl^2x}{16\,EI} + \frac{Px^3}{12\,EI}; \tag{3}$$

$$y = \frac{Px^3}{12\,EI} - \frac{Pl^2x}{16\,EI}. \tag{4}$$

Problems

1. A 4-inch by 6-inch wooden beam rests on supports 20 feet apart. When a load of 360 pounds is applied at the middle, the deflection at the middle is increased 1 inch. Find E.

Ans. 1,440,000 pounds per square inch.

2. A beam rests on two supports at a distance l apart and overhangs one support a distance a. How much is the overhanging end elevated when a load P is placed on the middle of the span? Solve by means of the moment triangle.

92. Beam Supported at the Ends with Uniformly Distributed Load.—The moment at a distance x from the left support is $\frac{wlx}{2} - \frac{wx^2}{2}$. These terms are shown separately in the lower diagram of Fig. 129. The second term is the same as that of a cantilever with a uniformly distributed load, and the first term is that of a cantilever with a concentrated load on the end, but opposite in sign, since the reaction is upward. From symmetry the tangent is horizontal at the middle, and the deflection at any point upward from this tangent may be calculated by combining the two cases of the cantilever with the proper signs.

If the distance $\frac{wx^2}{2}$ is measured downward from the line $\frac{wlx}{2}$, the remainder gives the parabola EFD representing the entire moment. In some cases it is convenient to use this single parabola in calculating the deflection, but generally it is better to use the separate figures of the lower diagram.

I. To find the deflection of the left end *upward* from the tangent at the middle by means of EFG (see Table XXIII),

$$EIy_{\max} = \frac{wl^3}{24} \times \frac{5}{8} \times \frac{l}{2} = \frac{5\,wl^4}{384}; \tag{1}$$

$$y_{\max} = \frac{5\,wl^4}{384\,EI} = \frac{5\,Wl^3}{384\,EI}. \quad \text{Formula XVIII.}$$

III. The deflection at some point, such as L, Fig. 129, from the line AB is computed by first finding its vertical distance y' from some tangent line AC and the distance of this tangent from AB

In Fig. 129 AC is tangent to the elastic line at the origin. The distance $NK = mx$, where m is the slope. $m = -\frac{CB}{l}$ if CB is measured upward. To get CB by area moments by means of the separate diagrams of Fig. 129, the moment is calculated with respect to the right end, for the triangle of base l, altitude $\frac{wl^2}{2}$, and area $\frac{wl^3}{4}$, and the parabola of base l, maximum ordinate $-\frac{wl^2}{2}$, and

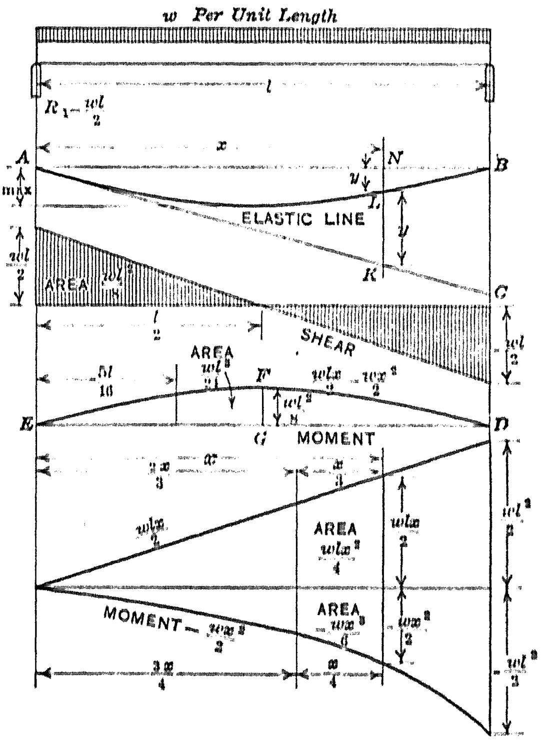


FIG. 129.—Beam supported at ends, distributed load.

area $\frac{wl^3}{6}$. The distances of the centers of gravity from the right end are $\frac{l}{3}$ and $\frac{l}{4}$ respectively.

$$EIy = \frac{wl^3}{4} \times \frac{l}{3} - \frac{wl^3}{6} \times \frac{l}{4} = \frac{wl^4}{24}; \tag{2}$$

$$y = CB = \frac{wl^4}{24\,EI}; \quad \text{slope} = m = -\frac{wl^3}{24}. \tag{3}$$

The distance $KL = y'$ is found in the same way using the triangle and parabola of base x,

$$EIy' = \frac{wlx^2}{4} \times \frac{x}{3} - \frac{wx^3}{6} \times \frac{x}{4} = \frac{wlx^3}{12} - \frac{wx^4}{24}; \tag{4}$$

$$y' = \frac{wlx^3}{12\,EI} - \frac{wx^4}{24\,EI}. \tag{5}$$

$$NK = mx = -\frac{wl^3x}{24\,EI}. \tag{6}$$

$$NK + y' = y = -\frac{wx}{24\,EI}(l^3 - 2\,lx^2 + x^3). \tag{7}$$

The deflection y is positive upward; NK is measured from N, and is therefore negative, while $KL(=y')$ is positive, being measured upward.

IV. To find the deflection of any point *upward* from the tangent at the middle,

$$M(x - x_1) = \frac{w}{2}(lx - x^2)(x - x_1). \tag{8}$$

$$EIy' = \frac{w}{2}\int (lx^2 - x^3 - lx_1x + x_1x^2)\,dx. \tag{9}$$

$$EIy' = \frac{w}{2}\left[\frac{lx^3}{3} - \frac{x^4}{4} - \frac{lx_1x^2}{2} + \frac{x_1x^3}{3}\right]_{x_1}^{\frac{l}{2}} \tag{10}$$

Substituting the limits and dropping the subscripts,

$$EIy' = \frac{5\,wl^4}{384} - \frac{wl^3x}{24} + \frac{wlx^3}{12} - \frac{wx^4}{24}. \tag{11}$$

To find y, the deflection downward from AB, the deflection at the middle, which is $-\frac{5\,wl^4}{384\,EI}$ is added to y' from (11), and the result is found to be the same as in (7).

Problems

1. Find CB by area moments using the diagram EFD of Fig. 129.
2. Find the slope of the tangent at the left end by differentiating equation (7).

93. Beam Supported at the Ends, Load at Any Point between Supports.—In Fig. 130 the moment diagram consists of the positive triangle of base l and the negative triangle of base b.

I. To find the deflection under the load from the line AB, the slope of AC is first determined. If $y' = BC$,

$$EIy' = \frac{Pbl}{2} \times \frac{l}{3} - \frac{Pb^2}{2} \times \frac{b}{3} = \frac{Pb}{6}(l^2 - b^2). \tag{1}$$

$$BC = \frac{Pb}{6\,EI}(l^2 - b^2); \quad m = -\frac{Pb}{6\,EIl}(l^2 - b^2). \tag{2}$$

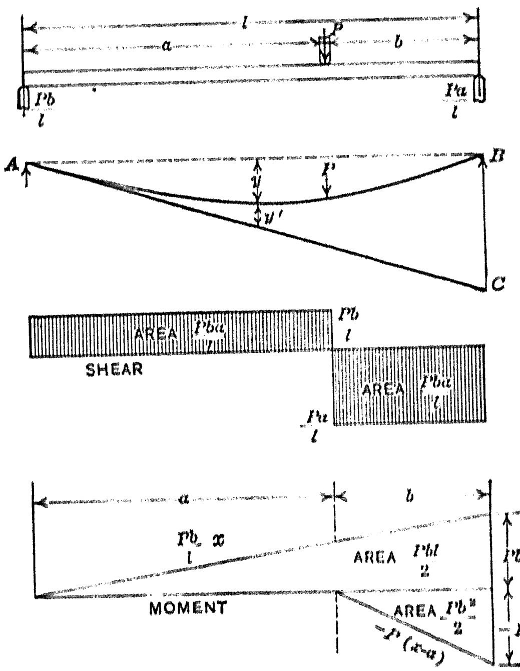


FIG. 130.—Beam with load at any point.

Under the load,

$$EIy' = \frac{Pba^2}{2\,l} \times \frac{a}{3} = \frac{Pba^3}{6\,l}. \tag{3}$$

$$y = ma + y' = -\frac{Pba}{6\,EIl}(l^2 - b^2) + \frac{Pba^3}{6\,EIl}; \tag{4}$$

$$y = -\frac{Pba}{6\,EIl}(l^2 - b^2 - a^2) = -\frac{Pa^2b^2}{3\,EIl}. \tag{5}$$

III. At a distance x from the left end (if x is less than a),

$$EIy' = \frac{Pbx^2}{2\,l} \times \frac{x}{3} = \frac{Pbx^3}{6\,l}.$$

$$y = -\frac{Pbx}{6\,EIl}(l^2 - b^2) + \frac{Pbx^3}{6\,EIl}; \tag{6}$$

$$y = -\frac{Pbx}{6\,EIl}(l^2 - b^2 - x^2). \tag{7}$$

To find the point of maximum deflection from the derivative of (7),

$$-l^2 + b^2 + 3x^2 = 0, \tag{8}$$

$$x = \sqrt{\frac{l^2 - b^2}{3}} = \sqrt{\frac{a^2 + 2ab}{3}}. \tag{9}$$

If a is greater than b, x is less than a and equation (9) gives the true location of the maximum deflection. If a is less than b, x is greater than a indicating that the maximum deflection is beyond the load. In that case x does not locate the point of maximum deflection, since equation (8) is not valid beyond the load.

94. The General Moment Equation.—Fig. 120 represents a beam with a load uniformly distributed which overhangs its supports. For the part between the supports the general moment equation may be applied. In this problem the moment is

$$M = -48 + 14x - \frac{x^2}{2}. \tag{1}$$

The first term is the horizontal straight line; the second term is the line AB, and the last term is the curve DF.

To find the slope of the tangent at the right support by means of the deflection of the left support B from the tangent line,

$$EIy = 6{,}300 \times 20 - 1{,}440 \times 15 - 4{,}500 \times 22.5 = 3{,}150,$$
$$3{,}150 \div 30 = 105,$$

which is the desired slope in terms of EI.

In a similar way the slope of the tangent at the left end is found to be $-\frac{255}{EI}$.

The deflection at the right end is found by combining the equations for a cantilever with uniformly distributed load and a cantilever with a load on the end, and subtracting the sum from the elevation of the tangent.

III. To find the deflection at any point between the supports by means of the diagram, it is convenient to first get the deflection from the tangent at the left support. The moment arm for each figure is the distance from its center of gravity to the ordinate at the *right*. This distance is $\frac{x}{3}$ for the triangle of base x, and $\frac{x}{4}$ for the parabola.

	Area	Arm	Moment
Triangle	$7\,x^2$	$\frac{x}{3}$	$\frac{7\,x^3}{3}$
Rectangle	$-48\,x$	$\frac{x}{2}$	$-24\,x^2$
Parabola	$\frac{x^3}{6}$	$\frac{x}{4}$	$-\frac{x^4}{24}$

The term due to the slope is $-255\,x$ so that the deflection from the horizontal line which joins the supports is given by

$$EIy = \frac{7\,x^3}{3} - \frac{x^4}{24} - 279\,x. \qquad (1)$$

95. Stiffness of Beams.—The stiffness of a beam is the reciprocal of the deflection. The stiffness of a beam may be defined as the load which will produce unit deflection. It is not customary to express stiffness in this way; it is generally used as a relative term.

In the expression for the maximum deflection of all the beams which we have considered, the terms E and I occur in the denominator. The stiffness of a beam varies directly as the modulus of elasticity and directly as the moment of inertia of its cross-section. The moment of inertia of a rectangular section varies as the cube of the depth, consequently the stiffness of a rectangular section varies in the same ratio. All the expressions for the maximum deflection contained the cube of the length in the numerator. The stiffness of beams of the same cross-section varies inversely as the cube of their length.

Problems

1. How does the stiffness of a 4-inch by 6-inch beam compare with that of a 4-inch by 4-inch beam of the same material?

2. How does the stiffness of a 4-inch by 6-inch beam with the 6-inch side vertical compare with that of the same beam with the 4-inch side vertical?

3. How does the stiffness of a 2-inch by 12-inch beam 15 feet long compare with that of a 2-inch by 8-inch beam 10 feet long? Which is the stronger?

CHAPTER IX

BEAMS WITH MORE THAN TWO SUPPORTS

96. Relation of Deflection to Stress.—In beams with *two* supports, including cantilevers, the moments and fiber stresses may be computed with no reference to the deflection. When there are more than two supports in the same plane, the problem of finding the reactions in an *absolutely rigid body* subjected to parallel forces is indeterminate. In elastic bodies (and all bodies are elastic) these reactions may be calculated if the equations of the elastic line are taken into account. Consequently a knowledge of these equations is indispensable for calculation of *stresses* except in the simplest cases. Since the unit stress is the most important factor from an engineering standpoint, this accounts for the prominence given to the equations of deflection.

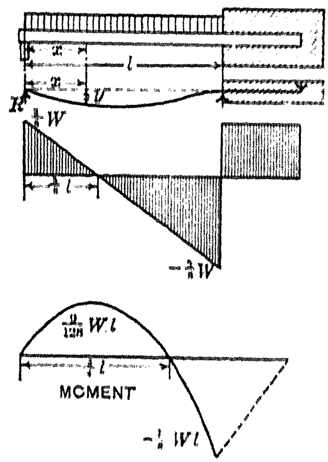


FIG. 131.—Beam fixed at one end and supported at the other.

97. Beam Fixed at One End and Supported at the Other.—Fig. 131 represents a beam fixed at the right end and supported at the left end, with the left end just touching the line which is tangent at the right end. The load is uniformly distributed.

If the left support is removed the beam becomes a cantilever, and the deflection downward at the left end (Formula XVII) is $\frac{wl^4}{8EI}$. The reaction R, regarded as a load on the end of a cantilever, must be sufficient to deflect the left end upward an equal amount.

$$\frac{Rl^3}{3EI} = \frac{wl^4}{8EI},\ R = \frac{3wl}{8} = \frac{3W}{8}. \tag{1}$$

With this reaction known,

$$M = \frac{3wlx}{8} - \frac{wx^2}{2}. \tag{2}$$

The equation of the elastic line is easily found by integration or by combination of the equations of a cantilever with a reaction at the end and a cantilever with a uniformly distributed load;

$$y = -\frac{w}{48\,EI}(2\,x^4 - 3\,lx^3 + l^3x) \qquad (3)$$

Problems

1. Draw shear and moment diagrams for beam fixed at one end and supported at the other. Find the moment at each dangerous section from the shear diagram and compare with the result from the equation of moments.

Ans. Moment at dangerous sections, $\frac{9\,Wl}{128}$, $-\frac{Wl}{8}$.

2. How does the greatest moment, numerically, compare with that of a beam supported at the ends?

3. Find the position of maximum deflection and the value of this maximum deflection.

Ans. Point of maximum deflection is 0.4215 l from the left support.

Differentiating equation (3) and equating to zero,

$$8\,x^3 - 9\,lx^2 + l^3 = 0.$$

Since the beam is horizontal at the wall, $x = l$ must satisfy this cubic. Division by the corresponding factor, $x - l$, gives a quadratic. Explain the meaning of the negative root.

98. Two Equal Spans, Uniformly Distributed Load.—Fig. 132 represents a continuous beam of two equal spans, each of

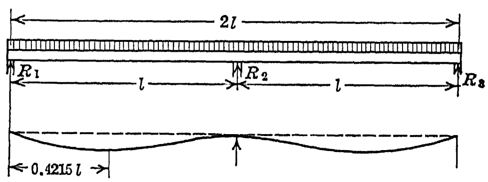


FIG. 132.—Beam with three supports.

length l. If the second support is removed, it becomes a beam of length $2\,l$ with two supports and the deflection at the middle is $\frac{5\,w(2\,l)^4}{384\,EI}$. The reaction at the second support, when the three supports are in the same straight line, must be equal to the load at the middle of a beam with two supports which would cause the same deflection.

$$\frac{R_2(2\,l)^3}{48\,EI} = \frac{5\,w(2\,l)^4}{384\,EI} \qquad (1)$$

$$R_2 = \frac{10\,wl}{8} \qquad (2)$$

The total load being $2\,wl$, it follows that $R_1 + R_3 = \frac{6\,wl}{8}$, and from symmetry $R_1 = R_3 = \frac{3\,wl}{8}$.

It will be noticed that the end reactions are the same as in Article 97. From symmetry it is evident that the beam is horizontal at the middle support, so that each half of the beam is equivalent to a beam fixed at one end and supported at the other.

Problems

1. A 6-inch by 8-inch wooden beam 20 feet long is supported at the ends and at the middle and carries a load, including its own weight of 480 pounds per foot. The end supports rest of footings 1 foot square. What should be the area of the footing for the middle post in order that the settlement of all shall be equal? What is the maximum fiber stress?

Ans. Fiber stress, 1,125 pounds per square inch.

2. In Problem 1 what will be the reaction of each post if the middle post settles 1.5 inches below the line of the others and E is 1,500,000 pounds per square inch? *Ans.* 2,800, 4,000, and 2,800 pounds.

3. Where will be the dangerous sections and what will be the maximum fiber stress at each in Problem 2.

Ans. 1,531 pounds per square inch at 70 inches from the ends. 750 pounds per square inch over the second support.

4. Solve Problems 2 and 3 if the end supports settle 1 inch below the second support.

5. What would be the maximum fiber stress if the beam is cut in two and merely rests on the second support?

99. Beam Fixed at One End, Supported at the Other, Load Concentrated.—The deflection at the end of a cantilever due to a load P at a distance a from the free end is given by equation (1), Article 78 and equation (2), Article 88;

$$y = -\frac{P}{6\,EI}(2\,l^3 - 3\,l^2a + a^3). \tag{1}$$

The end reaction which will prevent any deflection at the end must be equal and opposite to the load on the end which would produce an equal deflection, Fig. 133.

$$\frac{Rl^3}{3\,EI} = \frac{P}{6\,EI}(2\,l^3 - 3\,l^2a + a^3); \tag{2}$$

$$R = \frac{P}{2}\left(2 - \frac{3\,a}{l} + \frac{a^3}{l^3}\right). \tag{3}$$

The deflection at any point may be found by combining the formulas for a cantilever or by integrating:

$$EIy = -\frac{Rx}{6}(l^2 - x^2) + \frac{Px}{6l}(l - a)^3, \tag{4}$$

from $x = 0$ to $x = a$; and

$$EIy = -\frac{Rx}{6}(l^2 - x^2) + \frac{Px}{6l}(l - a)^3 - \frac{P}{6}(x - a)^3, \tag{5}$$

from $x = a$ to $x = l$.

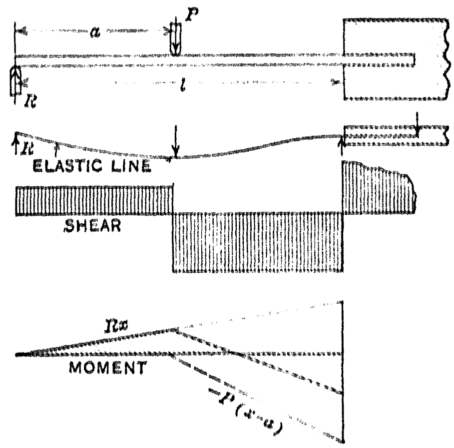


FIG. 133.—Beam fixed at one end and supported at other, load concentrated.

Problems

1. If $a = \frac{l}{2}$, find the reaction at the support and the moment under the load and at the fixed end.

Ans. $R = \frac{5P}{16}$; $M = \frac{5Pl}{32}$ under load; $M = -\frac{3Pl}{16}$ at fixed end.

2. A 2-inch by 1-inch wooden beam is securely clamped so that 8 feet projects as a cantilever. The free end rests on a platform scale. When a load of 20 pounds is placed 3 feet from the supported end what is the increase in the scale reading? *Ans.* 9.28 pounds.

3. In Problem 1 where is the point of counterflexure?

4. A beam 20 feet long is supported at the ends and at the middle and carries a distributed load of 40 pounds per foot and two symmetrically placed loads of 300 pounds each, 4 feet from the ends. Find the reaction at each support and the moment at each dangerous section.

Ans. $R_1 = R_3 = 279.6$ pounds; $R_2 = 840.8$ pounds; $M = 798.4$ foot-pounds at 4 feet; $M = -1{,}004$ foot-pounds at 10 feet.

100. Beam Fixed at Both Ends, Uniformly Loaded.—For uniformly distributed loading the general moment equation is

$$M = M_0 + V_0 x - \frac{wx^2}{2}. \tag{1}$$

At each end the deflection is zero and the tangent is horizontal. From symmetry it is evident that the shear at each end is one-half the total load, $V_0 = \frac{wl}{2} = \frac{W}{2}$.

(In this article and others following, the deflections will be calculated by the method of double integration and also by the method of area moments. The reader may omit either one of these methods.)

DOUBLE INTEGRATION

$$EI\frac{d^2y}{dx^2} = M_0 + \frac{wlx}{2} - \frac{wx^2}{2} \tag{2}$$

$$EI\frac{dy}{dx} = M_0 x + \frac{wlx^2}{4} - \frac{wx^3}{6} + [C_1 = 0]. \tag{3}$$

$\frac{dy}{dx} = 0$ when $x = 0$, hence $C_1 = 0$; $\frac{dy}{dx} = 0$ when $x = l$,

$$\text{from which } M_0 = -\frac{wl^2}{12} \tag{4}$$

If the deflection is desired,

$$EIy = -\frac{wl^2x^2}{24} + \frac{wlx^3}{12} - \frac{wx^4}{24} + [C_2 = 0]. \tag{5}$$

$$y = -\frac{wx^2}{24\,EI}(l - x)^2. \tag{6}$$

$$y_{\max} = \frac{wl^4}{384\,EI} = \frac{Wl^3}{384\,EI}. \tag{7}$$

AREA MOMENTS

To find the deflection at the left end from the tangent at the right end, Fig. 134,

$$EIy = 0 = \frac{wl^3}{4} \times \frac{2\,l}{3} + M_0 l \times \frac{l}{2} - \frac{wl^3}{6} \times \frac{3\,l}{4}; \tag{8}$$

$$\frac{wl^4}{6} + \frac{M_0 l^2}{2} - \frac{wl^4}{8} = 0, \quad M_0 = -\frac{wl^2}{12} = -\frac{Wl}{12} \tag{9}$$

To find the deflection at distance x from the left end, from the *tangent at the left end,* use the areas from the origin to the ordinate at a distance x, and take moments of these areas with respect to this ordinate.

$$EIy = \frac{wlx^2}{4} \times \frac{x}{3} - \frac{wl^2x}{12} \times \frac{x}{2} - \frac{wx^3}{6} \times \frac{x}{4}, \tag{10}$$

$$EIy = \frac{wlx^3}{12} - \frac{wl^2x^2}{24} - \frac{wx^4}{24}. \tag{11}$$

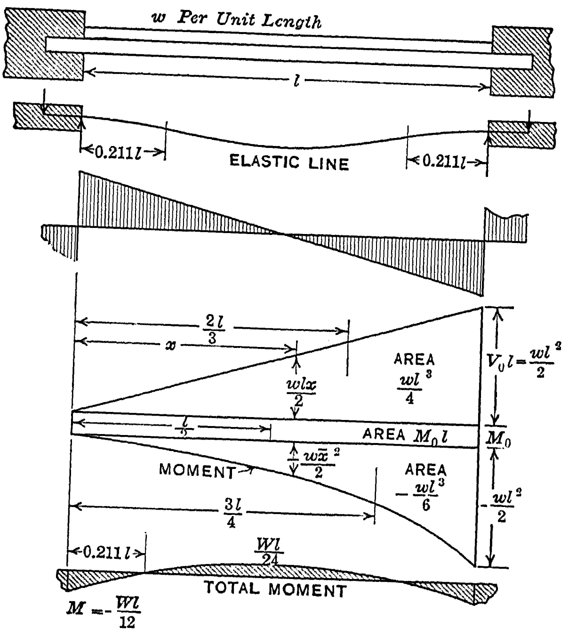


FIG. 134.—Beam fixed at both ends, uniformly loaded.

Problems

1. Show that the moment at the middle is one-half the moment at the ends.

2. Find the points of inflection. *Ans.* $0.211\,l$ and $0.789\,l$.

3. Show that the deflection at the middle is one-fifth as great as that of a beam supported at the ends.

In reinforced-concrete construction the beams are fixed to the columns. It is not possible to fix the end of a beam so that there is no change of slope, so that the maximum moment will be somewhat greater than $\frac{Wl}{12}$ and probably less than $\frac{Wl}{8}$. It is customary to use an intermediate figure and assume that the maximum moment is $\frac{Wl}{10}$.

101. Beam Fixed at Both Ends, Concentrated Load at Any Point.

DOUBLE INTEGRATION

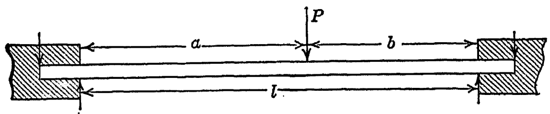


FIG. 135.—Beam fixed at ends with concentrated load.

From left end to load,

$$EI\frac{d^2y}{dx^2} = M_0 + V_0x. \tag{1}$$

$$EI\frac{dy}{dx} = M_0x + \frac{V_0x^2}{2} + [C_1 = 0]. \tag{3}$$

$$EIy = \frac{M_0x^2}{2} + \frac{V_0x^3}{6} + [C_2 = 0]. \tag{6}$$

From load to right end,

$$EI\frac{d^2y}{dx^2} = M_0 + V_0x - P(x - a). \tag{2}$$

$$EI\frac{dy}{dx} = M_0x + \frac{V_0x^2}{2} - \frac{P(x - a)^2}{2} + [C_3 = 0]. \tag{4}$$

$$0 = 2\,M_0l + V_0l^2 - P\,(l - a)^2. \tag{5}$$

$$EIy = \frac{M_0x^2}{2} + \frac{V_0x^3}{6} - \frac{P(x - a)^3}{6} + [C_4 = 0] \tag{7}$$

$$0 = 3\,M_0l^2 + V_0l^3 - P\,(l - a)^3. \tag{8}$$

From (5) and (8):

$$V_0 = \frac{3\,Pb^2}{l^2} - \frac{2\,Pb^3}{l^3} = \frac{Pb^2}{l^2}\left(3 - \frac{2\,b}{l}\right). \tag{9}$$

$$M_0 = -\frac{Pab^2}{l^2}. \tag{10}$$

These values substituted in (6) and (7) give the deflections to the left and right of the load, respectively.

AREA MOMENTS

To find the deflection of the left end from the tangent at the right end, Fig. 136,

$$EIy = 0 = M_0l \times \frac{l}{2} + \frac{V_0l^2}{2} \times \frac{2\,l}{3} - \frac{Pb^2}{2}\left(l - \frac{b}{3}\right), \tag{11}$$

$$3\,M_0l^2 + 2\,V_0l^3 - Pb^2(3\,l - b) = 0. \tag{12}$$

To find the deflection of the right end from the tangent at the left end,

$$EIy = 0 = M_0 l \times \frac{l}{2} + \frac{V_0 l^2}{2} \times \frac{l}{3} - \frac{Pb^2}{2} \times \frac{b}{3}, \tag{13}$$

$$3\,M_0 l^2 + V_0 l^3 - Pb^3 = 0. \tag{14}$$

From (12) and (14):

$$V_0 = \frac{Pb^2}{l^2}\left(3 - \frac{2\,b}{l}\right); \tag{15}$$

$$M_0 - = \frac{Pb^2}{l^2}(l - b) = -\frac{Pb^2 a}{l^2}. \tag{16}$$

To find the deflection at a distance x from the left end,

$$EIy = M_0 x \times \frac{x}{2} + \frac{V_0 x^2}{2} \times \frac{x}{3} = \frac{M_0 x^2}{2} + \frac{V_0 x^3}{6},$$

if x is not greater than a.

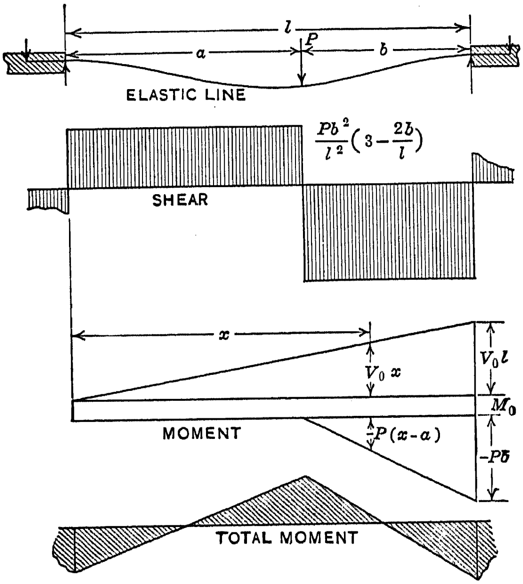


FIG. 136.—Beam fixed at ends.

Beyond the load, the term $\dfrac{P(x-a)^2}{2} \times \dfrac{(x-a)}{3}$ is subtracted,

$$EIy = \frac{M_0 x^2}{2} + \frac{V_0 x^3}{6} - \frac{P(x-a)^3}{6}.$$

Problems

1. If $a = \frac{l}{2}$, find the moment at the wall and under the load and find the shear at the wall.

$$Ans.\ M = -\frac{Pl}{8} \text{ at the wall; } M = \frac{Pl}{8} \text{ at the middle.}$$

2. In Problem 1 what is the deflection at the middle?

$$Ans.\ y_{\max} = -\frac{Pl^3}{192\,EI}.$$

3. How does the deflection and maximum stress in a beam fixed at the ends and loaded at the middle compare with those of a beam supported at the ends and loaded at the middle?

102. Theorem of Three Moments.—The methods of the preceding articles may be applied to any number of spans or to any number of concentrated loads. However, when it becomes

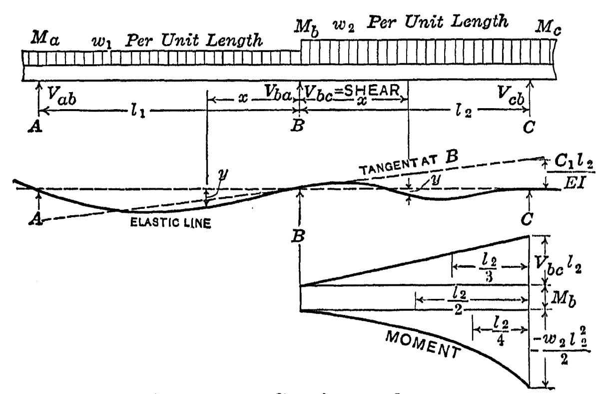


FIG. 137.—Continuous beam.

necessary to write more than two moment equations and solve for the corresponding constants, the work becomes laborious. When, as is usually the case, it is desired to find the moments, reactions, and shears, without getting the deflections, the *theorem of three moments* is of great use.

The theorem of three moments is an *algebraic equation* which expresses the relation of the moments at three successive supports of a continuous beam in terms of the length of the intervening spans and the loads which they carry. In Fig. 137, the moments over the supports are represented by M_a, M_b, M_c. The length of the span from support A to support B is l_1, and from B to C it is l_2. Fig. 137 represents a uniformly distributed load of w_1 pounds per unit length for the first span and w_2 pounds per unit length for the second span. The subscripts a, b, c,

represent the *order* from left to right and may be applied to *any three points* in *succession*. The same is true of the subscripts 1 and 2 applied to the spans and the unit loads.

The shear adjacent to B on the side toward C is designated by V_{bc}; on the side toward A by V_{ba}.

103. Theorem of Three Moments for Distributed Loads.—

DOUBLE INTEGRATION

Taking the origin of coördinates at the second support and considering the second span,

$$EI\frac{d^2y}{dx^2} = M_b + V_{bc}x - \frac{w_2x^2}{2}. \tag{1}$$

$$EI\frac{dy}{dx} = M_bx + \frac{V_{bc}x^2}{2} - \frac{w_2x^3}{6} + C_1. \tag{2}$$

The slope of the tangent at the support B is $\frac{C_1}{EI}$.

$$EIy = \frac{M_bx^2}{2} + \frac{V_{bc}x^3}{6} - \frac{w_2x^4}{24} + C_1x + [C_2 = 0]. \tag{3}$$

At $x = l_2$, $y = 0$,

$$\frac{M_bl_2}{2} + \frac{V_{bc}l_2^2}{6} - \frac{w_2l_2^3}{24} + C_1 = 0. \tag{4}$$

From the general moment equation,

$$M_c = M_b + V_{bc}l_2 - \frac{w_2l_2^2}{2}. \tag{5}$$

Substituting V_{bc} from (5) in (4)

$$2\,M_bl_2 + M_cl_2 + \frac{w_2l_2^3}{4} + 6\,C_1 = 0. \tag{6}$$

Using the span from A to B with the origin at B and x running from *right* to *left*,

$$M_al_1 + 2\,M_bl_1 + \frac{w_1l_1^3}{4} + 6\,C_3 = 0. \tag{7}$$

The slope of the tangent at B, going from right to left is $\frac{C_3}{EI}$, so that $C_3 = -C_1$, if the beam has uniform section throughout all of both spans. Adding (6) and (7),

$$M_al_1 + 2\,M_b(l_1 + l_2) + M_cl_2 = -\tfrac{1}{4}\,(w_1l_1^3 + w_2l_2^3). \tag{8}$$

Equation (8) is called the theorem of three moments for distributed loads.

AREA MOMENTS

Let m be the slope of the tangent to the beam at support B. This tangent is at a distance ml_1 below support A and at a distance ml_2 above support C.

To find the deflection at A,

$$EIy = M_a l_1 \times \frac{l_1}{2} + \frac{V_{ab} l_1^2}{2} \times \frac{2\,l_1}{3} - \frac{w_1 l_1^3}{6} \times \frac{3\,l_1}{4} - EIml_1 = 0,$$

$$12\,M_a l_1 + 8\,V_{ab} l_1^2 - 3\,w_1 l_1^3 - 24\,EIm = 0. \tag{9}$$

To find the deflection at C,

$$EIy = M_b l_2 \times \frac{l_2}{2} + \frac{V_{bc} l_2^2}{2} \times \frac{l_2}{3} - \frac{w_2 l_2^3}{6} \times \frac{l_2}{4} + EIml_2 = 0,$$

$$12\,M_b l_2 + 4\,V_{bc} l_2^2 - w_2 l_2^3 + 24\,EIm = 0. \tag{10}$$

From the general moment equation,

$$V_{ab} l_1 = M_b - M_a + \frac{w_1 l_1^2}{2}.$$

Substituting for V_{ab} in (9) and for V_{bc} in (10), and adding,

$$4\,M_a l_1 + 8\,M_b l_1 + w_1 l_1^3 - 24\,EIm = 0,$$
$$8\,M_b l_2 + 4\,M_c l_2 + w_2 l_2^3 + 24\,EIm = 0,$$
$$4\,M_a l_1 + 8\,M_b(l_1 + l_2) + 4\,M_c l_2 + w_1 l_1^3 + w_2 l_2^3 = 0.$$
$$M_a l_1 + 2\,M_b(l_1 + l_2) + M_c l_2 = -\tfrac{1}{4}\,(w_1 l_1^3 + w_2 l_2^3). \tag{11}$$

Equation (11) is the theorem of three moments for uniformly distributed loads on a beam of uniform section.

104. Calculation of Moments for Uniform Loading.—The theorem of three moments is an algebraic relation between the moments over any three successive supports of a beam of uniform section, provided these supports remain in a straight line when loaded. For a beam with three supports, one equation may be written by the theorem, and it is necessary to know two of the moments (or to have two other independent relations) in order to solve the problem. For four supports two equations are written, the first one for supports 1, 2, 3 in order as A, B, C, of the theorem, and the second for supports 2, 3, 4. For five supports three equations are written. In every case there are two more moments than there are independent equations.

If the spans are equal and the loads per unit length in the two successive spans are the same, the equation of three moments becomes:

$$M_a + 4\,M_b + M_c = -\frac{wl^2}{2}. \qquad \text{Formula XXII.}$$

Fig. 138 represents a beam with four supports and three equal spans with no overhang, with a load w per unit length. Representing the moments by the subscripts 1, 2, 3, 4, as they refer to particular supports as well as to the order of arrangement, the equations are:

$$M_1 + 4\,M_2 + M_3 = -\frac{wl^2}{2}, \tag{1}$$

$$M_2 + 4\,M_3 + M_4 = -\frac{wl^2}{2}. \tag{2}$$

If the beam does not overhang the end supports, $M_1 = 0$ and $M_4 = 0$. Solving the equations

$$M_2 = M_3 = -\frac{wl^2}{10}.$$

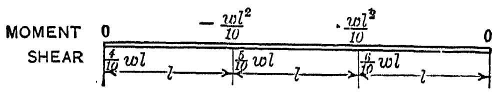


FIG. 138.—Beam of three equal spans.

Problems

1. Find the moments for two equal spans, with uniform loads on both, with no overhang at the end supports.

Ans. $M_1 = 0,\ M_2 = -\frac{wl^2}{8},\ M_3 = 0.$

2. Find the moments over the supports for four equal spans, with uniform loads on each and with no overhang.

Ans. $M_1 = 0,\ M_2 = -\frac{3\,wl^2}{28},\ M_3 = -\frac{wl^2}{14},\ M_4 = M_2,\ M_5 = 0.$

3. A uniformly loaded beam has two equal spans of length l and overhangs the left support 0.2 l and the right support 0.4 l. Find the moment at each support.

Ans. $M_1 = -0.02\,wl^2;\ M_2 = -0.10\,wl^2;\ M_3 = -0.08\,wl^2.$

4. A beam weighing w pounds per foot rests on four supports so as to make three 10-foot spans, and overhangs the left support 4 feet and the right support 2 feet. Find the moment at each support.

Ans. $M_1 = -8\,w;\ M_2 = -8\,w;\ M_3 = -10\,w;\ M_4 = -2\,w.$

5. A uniformly loaded beam 18 feet long is supported at the ends and 8 feet from the left end. Find the moment at the second support.

Ans. $-10.5\,w$ foot-pounds.

6. A shaft 30 feet long, weighing 10 pounds per foot, is supported 4 feet from the left end, 16 feet from the left end, and 6 feet from the right end and carries 60 pounds 1 foot from the left end. Find the moment over each support. *Ans.* $-260,\ -26,\ -180$ foot-pounds.

7. A uniformly loaded beam rests on three supports so as to have two equal spans with equal overhang on each end. What must be the ratio of overhang to span if the moments at all supports are the same?

Ans. Overhang 0.408 of the length of span.

If the beam ... support.

$M_1 = -200 \times 4 \times 2$

105. Calculation of Total Vertical Shear and Reactions.— If the moment at support B at a distance l_1 from support A which is taken as the origin, be written by means of the general moment equation,

$$M_b = M_a + V_{ab}l_1 - \frac{w_1 l_1^2}{2}, \tag{1}$$

$$V_{ab} = \frac{M_b - M_a}{l_1} + \frac{w_1 l_1}{2}.$$

V_{ab} is the shear just to the right of *any* support;

M_a is the moment at that support, and M_b at the next one;

$w_1 l_1$ is the total uniformly distributed load between these supports.

In the case of a beam with four supports and three equal spans with no overhang (Fig. 138) at the right of the left support,

$$V_{12} = \frac{-\frac{wl^2}{10} - 0}{l} + \frac{wl}{2} = 0.4\,wl = 0.4\,W.$$

At the right of the second support,

$$V_{23} = \frac{-\frac{wl^2}{10} + \frac{wl^2}{10}}{l} + \frac{wl}{2} = 0.5\,W.$$

In the same way, $V_{34} = 0.6\,wl = 0.6\,W$.

Fig. 138 gives the moments over the supports and the shears to the right of each support for the case of three equal spans.

Problems

1. Calculate the shear to the right of each support in Problem 2 of Article 104.

Ans. $V_{12} = \frac{11\,wl}{28}$, $V_{23} = \frac{15\,wl}{28}$, $V_{34} = \frac{13\,wl}{28}$, $V_{45} = \frac{17\,wl}{28}$.

2. Calculate the vertical shear to the right of each support in Problem 3 of Article 104.

Ans. 0.42 wl, 0.52 wl.

The shear at the left of any support is obtained from the shear at the right of the preceding support by subtracting the intervening load, according to the definition of vertical shear. The shear at the left of the second support in Fig. 138 is $-0.6\,wl$, at the left of the third support, $-0.5\,wl$, and at the left of the fourth support, $-0.4\,wl$.

The reaction at any support is computed by subtracting the shear at the left from the shear at the right of the support.

Fig. 139 gives moments, shears and reactions for a beam with four equal spans uniformly loaded with no overhang.

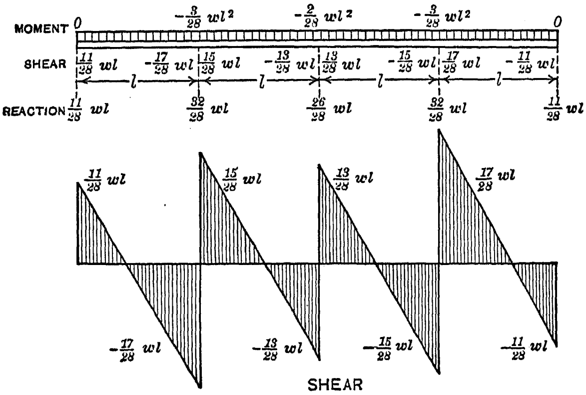


FIG. 139.—Beam of four spans.

Problems

3. Show that with three equal spans, uniformly loaded, the reactions are 0.4 wl, 1.1 wl, 1.1 wl, and 0.4 wl.

4. Calculate the reactions in Problem 3 of Article 104.
Ans. 0.62 wl, 1.10 wl, 0.88 wl.

5. From Fig. 139 locate the points of counterflexure.

6. A beam carrying a uniformly distributed load rests on three supports spaced 10 feet apart. How much should it overhang the outer supports in order that the reactions at all the supports shall be the same? *Ans.* 4.4 feet.

106. Theorem of Three Moments for Concentrated Loads.—Fig. 140 shows a continuous beam with a load P in the first span at a distance a from the *first* support, and a load Q in the second span at a distance c from the *third* support. If m is the slope of the tangent over the middle support, the deflection over the first support is

$$EIy = 0 = -ml_1 + \frac{M_a l_1^2}{2} + \frac{V_{ab} l_1^3}{3} - \frac{P(l_1 - a)^2}{2} \times \left[a + \frac{2\,(l_1 - a)}{3}\right]; \quad (1)$$

$$0 = -6m + 3M_a l_1 + 2V_{ab} l_1^2 - \frac{2\,P(l_1 - a)^3}{l_1} - \frac{3\,Pa(l_1 - a)^2}{l_1}. \quad (2)$$

From the general moment equation,

$$M_b = M_a + V_{ab}l_1 - P(l_1 - a),$$
$$2\,V_{ab}l_1^2 = 2\,M_b l_1 - 2\,M_a l_1 + 2\,P(l_1 - a),$$

which substituted in (2) gives,

$$0 = -6\,m + M_a l_1 + 2\,M_b l_1 + \frac{Pa\,(l_1^2 - a^2)}{l_1}. \qquad (3)$$

The deflection over the third support is found in a similar way. Expressing the moment in the *second* span in terms of the shear at the *third* support.

$$0 = 6\,m + M_c l_2 + 2\,M_b l_2 + \frac{Qc(l_2^2 - c^2)}{l_2}. \qquad (4)$$

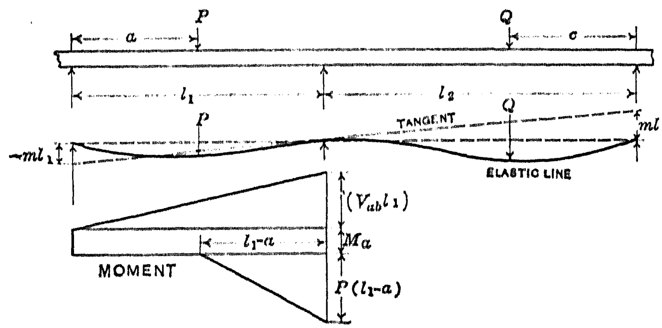


FIG. 140.—Continuous beam with concentrated loads.

Adding (3) and (4), the theorem of three moments for single concentrated loads is,

$$M_a l_1 + 2\,M_b(l_1 + l_2) + M_c l_2 = -\frac{Pa(l_1^2 - a^2)}{l_1} - \frac{Qc(l_2^2 - c^2)}{l_2}. \qquad (5)$$

The first member of (5) is the same as in the theorem for uniformly distributed loads. For distributed and concentrated loading combined it is necessary only to add the second member of (8), Article 103, to the second member of (5) above.

If there are more than one concentrated load in any span, it is taken care of by an additional term in the second member of the equation. For uniformly distributed loads combined with more than one concentrated load, the general equation of three moments is,

$$M_a l_1 + 2\,M_b(l_1 + l_2) + M_c l_2 = -\frac{w_1 l_1^3}{4} - \frac{w_2 l_2^3}{4} - \sum\frac{Pa\,(l_1^2 - a^2)}{l_1} - \sum\frac{Qc\,(l_2^2 - c^2)}{l_2},$$

where $\Sigma Pa(l_1^2 - a^2)$ is the sum of the terms $P_1a_1(l_1^2 - a_1^2) + P_2a_2(l_1^2 - a_2^2) +$, etc. P_1 is the first load at a distance a_1 from the support A; P_2 is the second load at a distance a_a from A, etc.

In the second span, Q_1 is the load at a distance c_1 from support C; Q_2 is a load at a distance c_2 from C; and so forth.

Problems

1. A beam of length $2\,l$ is supported at the ends and at the middle and carries a load P at a distance $\frac{l}{3}$ from the left end, and an equal load at the same distance from the right end. Find the moment over the middle support and the reaction at each support.

$$Ans.\ M = -\frac{4Pl}{27};\ R_1 = R_3 = \frac{14P}{27};\ R_2 = \frac{26P}{27}.$$

2. A line shaft 22 feet long, weighing 10 pounds per foot, is supported at the ends and 10 feet from the left end, and carries 60 pounds 4 feet from the left end, 40 pounds 7 feet from the left end, and 50 pounds 7 feet from the right end. Find the moment over the second support and the reactions.

Ans. $M_2 = -296.25$ foot-pounds; $R_1 = 68.4$ pounds.

107. Deflection Due to Moments not Parallel to Principal Axis of Inertia.—When the bending moment was not parallel to one of the principal axes of inertia it was found necessary to resolve the moment or the forces parallel to these axes before calculating the fiber stress.

In the same way, to find the deflection, the forces must be resolved into components and the deflections calculated parallel to each of these two axes. The resultant deflection at any point is the vector sum of the components.

Example

A 2-inch by 3-inch wooden cantilever, 10 feet long, has the 3-inch faces at an angle of 35 degrees with the horizontal. Find the magnitude and direction of the deflection at the end due to a load of 20 pounds on the end, if E is 1,200,000 pounds per square inch.

The components of the load are 20 cos 35 degrees and 20 sin 35 degrees. The corresponding moments of inertia are 2 inches^4 and 4.5 inches^4, respectively. The deflection perpendicular to the 3-inch faces is $4.8 \times 0.8192 = 3.932$ inches.

The deflection parallel to the 3-inch faces is

$$\frac{32}{15} \times 0.5736 = 1.224 \text{ inches.}$$

The angle ϕ which the resultant deflection makes with the 2-inch faces is given by

$$\tan\phi = \frac{1.224}{3.932} = \frac{32 \sin 35°}{15 \times 4.8 \cos 35°} = \frac{2}{4.5}\tan 35° = 0.3112.$$

$$\phi = 17°\ 17'.$$

Resultant deflection $= 3.932 \sec\phi = 4.118$ inches, at 17 degrees 43 minutes with the vertical.

Problems

1. A 3-inch by 4-inch wooden cantilever 5 feet long, is placed with one diagonal horizontal. Find the deflection at the end due to a load of 90 pounds on the end, if E is 1,500,000 pounds per square inch.

2. Two 6-inch by 4-inch by 1-inch angles are placed with the 6-inch legs vertical so as to form parallel cantilevers 10 feet in length. If the 4-inch legs are in opposite directions and away from each other, and E is 29,000,000 pounds per square inch, how much will the ends separate when a load of 600 pounds is placed on each cantilever?

108. Deflection from Moments in More than One Plane.— When the forces acting on a beam are not all parallel to one plane which passes through the beam, it is necessary to resolve the forces into components parallel to two axes which are perpendicular to each other and to the length of the beam. If the beam is circular, square, or of any other section for which the moment of inertia is the same in every direction, these axes may be taken in any convenient way. For all other sections the resolutions must be made parallel to one of the principal axes of inertia. The two components of the deflection at any point are calculated separately, and the resultant deflection found from their vector sum.

Example

A 3-inch solid shaft, weighing 24 pounds per foot, is 10 feet long and is supported at the ends. A pulley weighing 160 pounds is 3 feet from the left end, and is subjected to a pull of 400 pounds 30 degrees below the horizontal in a plane perpendicular to the length of the shaft. Find the deflection at the pulley, if E is 29,000,000 pounds per square inch.

Resolving vertically, the total vertical load at the pulley is 360 pounds. The horizontal pull is 346.4 pounds. The deflections at 36 inches from one end are:

From concentrated load of 360 pounds	0.0793 inch.
From load of 2 pounds per inch	0.0381 inch.
Total vertical deflection	0.1174 inch.

The horizontal deflection from load of 346.4 pounds is 0.0763 inch.

Problem

1. A 10-inch 15-pound channel 20 feet long is supported at the ends with the web inclined 20 degrees to the vertical. It carries a vertical load of 300 pounds per foot and a load of 400 pounds per foot perpendicular to the flange. Find the deflection and fiber stress at the middle.

CHAPTER X

SHEAR IN BEAMS

109. Direction of Shear.—The total vertical shear in a beam is calculated by the methods of Article 53, but this gives no information in regard to the distribution of the shearing stress in the section. In Article 30 it was shown that shearing stresses occur in pairs, and that a small block subjected to shearing stress of given intensity along two parallel faces is subjected to a shearing stress of the same intensity along two other faces at right angles to these.

Fig. 141, I, represents a beam made by placing one plank on top of another. Fig. 141, II, is the same beam under load, provided that the planks are held from slipping with reference to each other by being glued or bolted together to form a single beam. If the planks are free to move, they take the form III, in which the upper plank is moved outward over the lower one at each end. Consider a small block B in the upper portion of the lower plank.

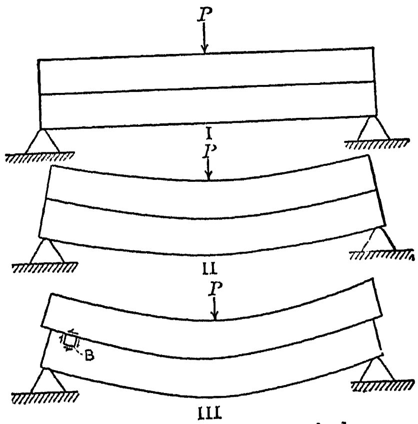


FIG. 141.—Horizontal shear in beams.

The plank above this block has been displaced to the left. If they were glued together, the upper plank would have exerted a horizontal shearing stress upon the upper surface of the block. To prevent rotation there must be a vertical shear upward at the left side. The actual shearing stresses upon this block from the surrounding material, if the upper plank were glued to the lower, would take the directions of the arrows.

The shear at the left of the block is vertically upward, which is the direction of the external shear. If a block were taken to the right of the load P, it would be found that the shear on its left side is vertically downward, which is the direction of the vertical

shear in that part of the beam. One of the planks of Fig. 141 may be thicker than the other, but the *direction* of the shear will remain the same.

110. Intensity of Shearing Stress.—Fig. 142 represents a part of a beam subjected to bending moment and vertical shear. A small block is shown extending across the beam between vertical planes dx apart and reaching from the top of the beam to a horizontal plane at a distance v_3 from the neutral surface. Two

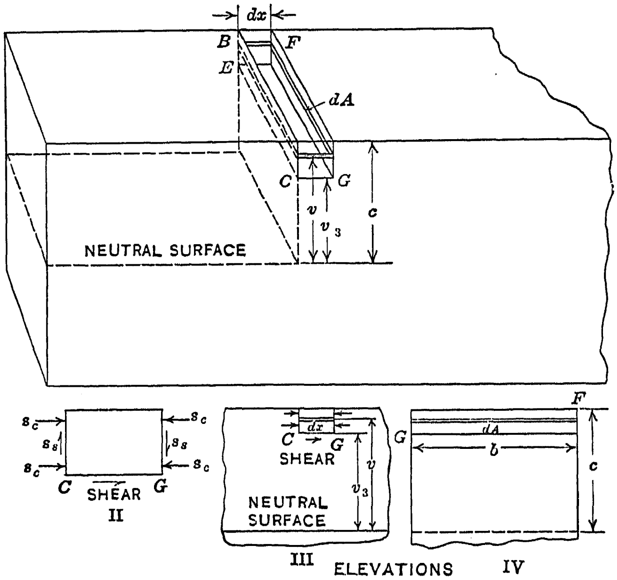


Fig. 142.—Horizontal shear in rectangular section.

elevations of this block and the adjoining parts of the beam are shown in Fig. 142, III and IV, and an enlarged elevation of the block in Fig. 142, II. The block is in equilibrium under the action of the compressive stress on the ends (the rectangles whose diagonals are CB and GF), the vertical shearing stress on the same surfaces, the horizontal shear from the material below (on the rectangle GE), and vertical compression or tension across the base.

Consider an element dA in the left end of this block. The unit compressive stress on this area is $\frac{M_1 v}{I_1}$ where M_1 is the bending moment at the section, and I_1 is the moment of inertia of the

entire cross-section of the beam with respect to the neutral axis. The total compression on the left end of the block is integral of the unit stress over the surface of the end.

$$\text{Total compression on left end} = \frac{M_1}{I_1}\int_{v_3}^{c} v\,dA. \tag{1}$$

$$\text{Total compression on right end} = \frac{M_2}{I_2}\int_{v_3}^{c} v\,dA. \tag{2}$$

The resultant horizontal push on the block in the direction of the length of the beam is the difference of these integrals (1) and (2). If the section of the beam is uniform $I_1 = I_2$ and v_3 and c_2 are the same for both expressions. The resultant horizontal pull (or push) becomes:

$$\text{Resultant force} = \frac{M_2 - M_1}{I}\int_{v_3}^{c} v\,dA. \tag{3}$$

This resultant horizontal force must be balanced by the horizontal shear at the bottom of the block. If the breadth CE at the bottom of the block is b, the total area in horizontal shear is $b\,dx$, and the total shear is $s_s b\,dx$. Equating these forces:

$$s_s b\,dx = \frac{M_2 - M_1}{I}\int_{v_3}^{c} v\,dA; \tag{4}$$

$$s_s = \frac{M_2 - M_1}{Ib\,dx}\int_{v_3}^{c} v\,dA. \tag{5}$$

Since $M_2 - M_1$ is equal to dM,

$$\frac{M_2 - M_1}{dx} = \frac{dM}{dx} = V, \tag{6}$$

where V is the total vertical shear.

$$s_s = \frac{V}{Ib}\int_{v_3}^{c} v\,dA, \tag{7}$$

where s_s equals the unit horizontal shear at a distance v_3 from the neutral axis and also equals the unit vertical shear at the same place. The term $\int_{v_3}^{c} v\,dA$ is the moment of the area of the end of the block with respect to the neutral axis.

$$\bar{v} = \frac{\int_{v_3}^{c} v\,dA}{A};\quad \int_{v_3}^{c} v\,dA = \bar{v}A. \tag{8}$$

When the area and location of the center of gravity of the portion of the plane section above the line CE are known, the integral

may be replaced by the equivalent expression of (8). Equation (7) then becomes

$$s_s = \frac{V}{Ib}\bar{v}A. \qquad \text{Formula XXIII.}$$

The theory above has been derived for compressive stress. It applies as well to tension. It assumes that the unit stress varies as the distance from the neutral surface. It is valid, therefore, only when the tensile and compressive stresses in the outer fibers are within the proportional elastic limit. But the greatest shear is usually at sections where the bending moment is small and the bending stress is below the elastic limit, so that no correction is necessary for this reason.

Formula XXIII gives the unit horizontal shearing stress in the beam. The unit vertical shearing stress has been shown to be the

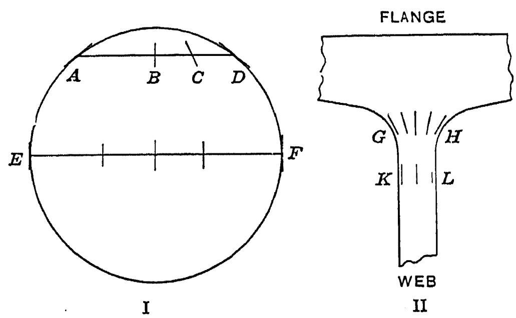


FIG. 143.—Shear in curved sections.

same as the unit horizontal shearing stress in a block *subjected to shear alone*. In a beam where the shear is combined with tension or compression, it may be shown by means of the moments of the forces acting on the block BG that at a given point these two unit shearing stresses are practically equivalent. The unit horizontal shearing stress increases from the outer fibers to the neutral surface, and the unit vertical shearing stress changes in the same way.

Formula XXIII gives the average unit shearing stress in the plane GE of Fig. 142. If the section is not rectangular, the unit stress may not be uniform in a horizontal surface. Fig. 143, I, is a circular section. AD is the trace of a horizontal plane. The short lines are the traces of planes in which the shear is transmitted from one side of AD to the other. At the middle the shear

$S = \frac{MC}{I}$. is for bending stress.

is transmitted from a filament above the plane to one directly below. At A and D the shear is transmitted from a filament in the surface to another filament in the surface, and here the short lines are tangential. At the diameter EF the shear is transmitted from one filament to another directly below, and it is customary to *assume* that the distribution is uniform.

Fig. 143, II, is part of an I-beam section. At the plane where the web joins the flange, there must be a great difference in the intensity of the shearing stress. At KL, at some little distance down the web, the shearing stress becomes practically uniform over the section.

Example

Find the horizontal unit shearing stress in a 6-inch by 8-inch rectangular section, at a plane 2 inches from the top, if the total vertical shear is 3,840 pounds.

$$\frac{V}{Ib} = \frac{3{,}840}{256 \times 6} = 2.5; \quad \bar{v}A = 3 \times 12 = 36;$$

$$s_s = 2.5 \times 36 = 90 \text{ pounds per square inch.}$$

Problems

1. In the example above, find the unit shearing stress at 1 inch from the top, at 3 inches from the top, and at the neutral surface.

Ans. 52.5; 112.5, 120 pounds per square inch.

2. In Problem 1 find the average vertical unit shearing stress by dividing the total vertical shear by the area. *Ans.* 80 pounds per square inch.

3. Show algebraically that in beams of rectangular section the average unit shearing stress is two-thirds as great as the unit shearing stress at the neutral surface.

4. A 4-inch by 6-inch wooden beam, weighing 6 pounds per foot, is 10 feet long and is supported at the ends. It carries a load of 600 pounds 4 feet from one end. By means of the result of Problem 3 find the unit shearing stress at the neutral surface at the section at which the total vertical shear is the greatest. *Ans.* 24.375 pounds per square inch.

5. In a beam of solid circular section, what is the ratio of the unit shearing stress at the neutral surface to the average unit shearing stress, assuming that the unit stress is uniform? *Ans.* 4 : 3.

6. Using the allowable unit shearing stress adopted by the American Railway Engineering and Maintenance of Way Association (see handbook) find the maximum load which may be placed at the middle of a short beam of Douglas fir, which is 6 inches by 10 inches, and is supported at the ends.

Ans. 8,800 pounds.

7. In Problem 6 what is the maximum distance between the supports in order that the bending stress shall not exceed the allowable value, and what must be the area of the supports at the ends?

$\bar{v}$ is the distance from the neutral axis to the center of cut section.

8.* A 7-inch by 14-inch beam of long-leaf yellow pine, placed on supports 13 feet 6 inches apart, was subjected to equal loads at points 4 feet 6 inches from the supports. When the total load was 57,500 pounds, the beam failed by shear at the neutral axis at one end. Find the ultimate shearing strength of this timber parallel to the grain. Compare the result with the figures given by the United States Department of Agriculture (see handbook).

Ans. 440 pounds per square inch.

9.* A 7-inch by 16-inch beam of Douglas fir, supported at points 13 feet 6 inches apart and loaded at the third points with equal loads, failed by shear when the total load was 45,000 pounds. Find the ultimate shearing strength of this timber parallel to the grain. *Ans.* 301 pounds per square inch.

10. Timber having an allowable unit shearing stress, parallel to the grain, of 100 pounds per square inch, and an allowable bending stress of 1,000 pounds per square inch, is used for beams supported at the ends and loaded at the middle. Below what length will the shear determine the load in a 4-inch by 6-inch beam?

The total vertical shear at either end is,

$$V = 24 \times \tfrac{2}{3} \times 100 = 1{,}600 \text{ pounds.}$$

The maximum moment under the load is,

$$M = 1{,}000 \times \frac{bd^2}{6} = 24{,}000 \text{ inch-pounds.}$$

$$1{,}600 \times \frac{l}{2} = 24{,}000.$$

$$\frac{l}{2} = 15 \text{ inches, } l = 30 \text{ inches.}$$

11. The timber of Problem 10 is used to support a load which is uniformly distributed. Below what length will the shear determine the load in the case of a 4-inch by 6-inch beam? Solve also for a 6-inch by 10-inch beam, and for an 8-inch by 6-inch beam. *Ans.* 5 feet, 8 feet 4 inches, 5 feet.

12. In Problems 8 and 9 what was the maximum bending stress?

111. Shearing Stress in I-beams.—It is customary to calculate the unit shearing stress in the web of an I-beam by dividing the total vertical shear by the area of cross-section of the web regarded as extending the entire depth of the beam. If t is the thickness of the web and d is the depth of the beam it is assumed that

$$\text{Average unit shearing stress} = \frac{\text{total vertical shear}}{td}.$$

In a 12-inch 31.5-pound I-beam, Fig. 144, the thickness of the web is 0.35 inch; the area td is 4.2 square inches, and the average unit shearing stress, as computed by this method, is $0.238\, V$.

Calculating the unit shearing stress in a 12-inch, 31.5-pound I-beam at the neutral surface by Formula XXIII,

* Problems 8 and 9 are from tests made by Prof. A. N. Talbot, described in *Bulletin* No. 41 of the Engineering Experiment Station of The University of Illinois.

	A	$\bar{v}$	$\bar{v}A$
Horizontal rectangle	1.750	5.825	10.194
Two triangles	0.907	5.520	5.006
Vertical rectangle	1.977	2.825	5.585
Total			20.785

$$\frac{V}{Ib}\bar{v}A = \frac{20.785\ V}{215.8 \times 0.35} = 0.275\ V.$$

At 5 inches from the neutral surface,

$$\bar{v}A = 20.785 - 5 \times 0.35 \times 2.5 = 20.785 - 4.375 = 16.409.$$

$$s_s = \frac{16.409V}{215.8 \times 0.35} = 0.217V.$$

The average of 0.275 V and 0.217 V is 0.246 V which differs very little from 0.238 V. It is evident that the method of calculating average unit shear in an I-beam section gives a result which is practically correct.

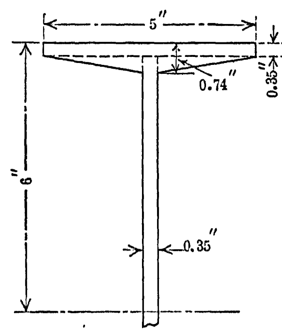


FIG. 144.—I-beam section.

Problem

Calculate the unit shearing stress in terms of the total shear in the web of a 10-inch 25-pound I-beam at the neutral surface and at the bottom of the flange.

Ans. $s_s = 0.368\ V$ at the neutral surface.
$s_s = 0.291\ V$ at the bottom of the flange.

112. Relation of Shearing Stress to Stress-distribution Diagram.—The unit tensile or compressive stress in a beam, provided the elastic limit is not exceeded, varies as the distance from the neutral surface. At a distance v from the neutral axis the unit stress is kv (where k is a constant) and the total stress on an area dA is $kvdA$. The total stress on an area extending from the plane at a distance v_3 from the neutral axis to the top of the beam is $k\int vdA = k\bar{v}A$. Comparing with equation (6) of Article 110 or with Formula XXIII, it is evident that the unit longitudinal shearing stress at any surface is proportional to the total tension or compression above or below that surface. The total tension or compression above a given surface is represented by the area of the stress-distribution diagram above that surface, so that the stress-distribution diagram shows the variation of the unit shearing stress in the section.

Fig. 145 is the stress-distribution diagram for a rectangular section. The area between the neutral axis and a line at one-fourth the depth above the axis is one-fourth of the total area above the axis. The area above this line, then, is three-fourths of the area of the entire triangle, and the unit shearing stress is three-fourths as great as that at the neutral surface.

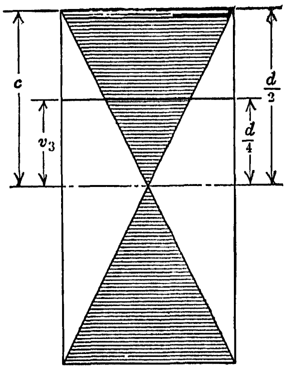


FIG. 145.—Distribution diagram.

In an I-beam most of the shaded area in the stress-distribution diagram is in the flange. The small shaded area in the web measures the difference between the shearing stress at the neutral axis and that at the bottom of the flange.

Example

A 4-inch by 10-inch rectangular beam is subjected to a total vertical shear of 2,000 pounds. Find the unit shearing stress at each inch above the neutral axis by means of the stress distribution diagram.

The average unit shearing stress is 50 pounds per square inch, and the unit shearing stress at the neutral surface is $\frac{3}{2} \times 50 = 75$ pounds per square inch. The area of the stress distribution triangle above the neutral axis is 10 square inches, and the area of the similar triangle below the 1-inch line is $\frac{1}{25}$ as great. The area of the diagram above the 1-inch line is $\frac{24}{25}$ of that of the total triangle. The unit shearing stress at 1 inch from the neutral axis is $\frac{24}{25} \times 75 = 72$ pounds per square inch. At 2 inches the unit stress is $\frac{4}{25} \times 75 = 12$ pounds less than at the neutral surface.

Problem

The unit shearing stress in a 5-inch by 12-inch beam at the neutral surface is 72 pounds per square inch. What is the unit shearing stress at each inch above or below that surface? Solve without writing.

113. Failure of Beams.—The nature of the failure in a beam depends principally upon the relative ultimate strength of the material in the different directions and the value of the different maximum stresses. In a beam which is short relative to its depth, the unit tensile and compressive stresses at the dangerous section are small compared with the unit shearing stress at the neutral surface at the ends. Owing to the fact that timber has a small shearing strength parallel to the grain, such a beam, if made of timber, will usually fail by shear. Fig. 146 shows 4 wooden

beams each about 40 inches long. The upper beam is a yellow pine beam glued to a white pine beam. The total depth was 3.80 inches and breadth 1.57 inches. The beam was supported at points 36 inches apart and loaded at the third points; this beam failed by longitudinal shear at one end when the total load was 1,950 pounds. The failure followed the glued surface but began in the white pine.

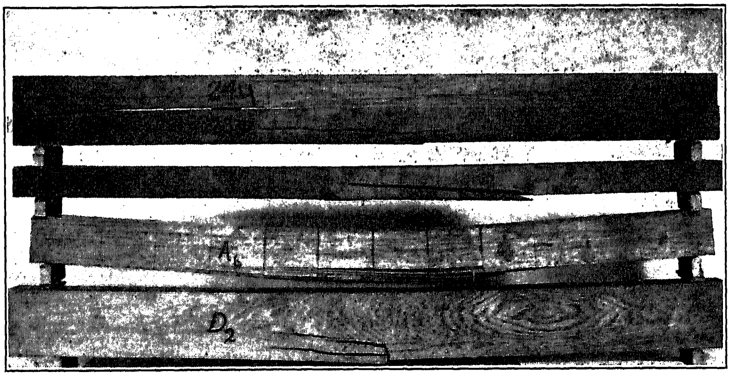


FIG. 146.—Failure of timber beams.

Beams of brittle material, such as cast iron, hard steel, stone, or concrete which is not reinforced, fail by tension. Beams of soft steel fail by buckling on the compression side or by buckling of the web in the case of I-beam sections.

114. Deflection Due to Shear.—The deflection of a beam due to shear is sometimes taken into account. If the unit shearing stress across a section were constant the shearing deformation in a length dx would be $\frac{s_s}{E_s}dx$, and the total deflection in a length l would be

$$y_s = \frac{1}{E_s}\int_0^l s_s dx. \qquad (1)$$

If s_s is constant throughout the length l this becomes,

$$y_s = \frac{s_s l}{E_s}. \qquad (2)$$

In an I-beam section the unit shearing stress is *assumed* to be constant and the equations above apply to give an *approximate* result.

Example

Find the deflection at the middle of a 10-inch 25-pound I-beam, supported at points 12 inches apart, due to a load of 49,600 pounds at the middle of the span. $E_s = 12{,}000{,}000$ lb./in.^{2}

The vertical shear is 24,800 pounds, the web area is 3.1 square inches making $s_s = 8{,}000$ pounds per square inch. Regarding the middle as fixed, the shear of either end upward is

$$y_s = \frac{8{,}000 \times 6}{12{,}000{,}000} = 0.004 \text{ inch.}$$

The deflection due to bending is

$$y = \frac{49{,}600 \times 12^3}{48 \times 29{,}000{,}000 \times 122.1} = 0.005 \text{ inch,}$$

so that in this extreme case the deflection due to shear is greater than that due to bending. If the beam were made twice as long, the bending deflection would be eight times as great while the shear deflection would be only twice as great. For beams of any considerable length relative to their cross-section the deflection due to shear may be neglected.

The shearing stress in a beam is not uniformly distributed in the cross-section. It is possible to calculate the true deflection due to shear in sections for which the true shear distribution is known. It will be shown in Article 162 that the deflection of a beam of rectangular section may be calculated by multiplying the average unit shearing stress by the factor 1.2.

Example

A steel cantilever 2 inches square and 40 inches long has a load of 240 pounds on the free end. Find the deflection of the end due to the shear caused by this load, if E_s is 12,000,000 pounds per square inch.

The average unit shearing stress is 60 pounds per square inch,

$$y_s = \frac{1.2 \times 60 \times 40}{12{,}000{,}000} = 0.00024 \text{ inch.}$$

If E is 30,000,000 the deflection due to bending is 0.128 inch, so that the deflection due to shear is relatively negligible. If the load were made four times as great and the length reduced to 10 inches, y_s would remain 0.00024 inch but y would become only 0.008 inch. In this case the deflection due to shear is relatively important.

Problems

1. A 2-inch by 3-inch steel beam rests on supports 12 inches apart and carries a load of 12,000 pounds midway between the supports. If E_s is 12,000,000 and E is 30,000,000 pounds per square inch, find the deflection due to shear and due to bending. *Ans.* $y_s = 0.0006$ inch, $y = 0.0032$ inch.

2. The beam of Problem 1 carries a distributed load of 1,600 pounds per inch. Find the deflection due to shear and to bending.

Ans. $y_s = 0.00048$ inch; $y = 0.0032$ inch.

CHAPTER XI

SPECIAL BEAMS

115. Beams of Constant Strength.—A beam of "constant strength" is one in which the section modulus varies as the bending moment, so that the bending stress in the outer fibers is the same at all sections. To design such a beam, the moment is written and equated to the product of the allowable unit stress multiplied by the section modulus. From this the section modulus is calculated and the dimensions determined in accordance with the other conditions of the design.

116. Cantilever with Load on the End.—With the origin at the free end of the cantilever, the moment at a distance x from the free end is Px. If S is the allowable unit bending stress,

$$Px = S \text{ multiplied by the section modulus.}$$

For a rectangular section the section modulus is $\frac{bd^2}{6}$, and

$$Px = \frac{Sbd^2}{6}.$$

Problems

1. A cantilever of constant strength, with the load on the end, is of rectangular section of constant depth 6 inches. The allowable fiber stress is 800 pounds per square inch. Find the equation for the breadth.

Ans. $b = \frac{Px}{4{,}800}$.

2. A cantilever beam of constant strength and rectangular section has a constant breadth b. If the allowable unit stress is S, find the expression for the depth for a load on the free end. *Ans.* $d^2 = \frac{6\,Px}{Sb}$.

3. A cantilever of constant strength with load of 600 pounds at the free end is 4 inches wide throughout. The section is rectangular. The allowable fiber stress is 1,200 pounds per square inch. If the length is 60 inches from the load to the fixed point, find the depth at each 10 inches.

Ans.							
	Position:	10	20	30	40	50	60 inches.
	Depth:	2.74	3.87	4.74	5.48	6.12	6.71 inches.

4. A cantilever 5 feet long carries a load of 800 pounds at the free end. The section is a rectangle with the depth twice the breadth. The allowable stress is 1,000 pounds per square inch. Find the depth at each 10 inches.

Ans.							
	Position:	10	20	30	40	50	60 inches.
	Depth:	4.58	5.77	6.60	7.27	7.83	8.32 inches.

5. A cantilever of constant strength is 5 feet long and carries a load at the free end. The breadth is constant and the depth at the wall is 8 inches. Find the depth at each foot. *Ans.* 3.58, 5.06, 6.20, 7.16, 8.0 inches.

6. The beam of Problem 5 is of a square section and is 8 inches square at the wall. Find the dimensions at each 10 inches.

Fig. 147 shows some cantilevers of constant strength and rectangular section. Fig. 147, I, is a beam of constant depth. The breadth varies as x—the equation of a straight line. The plan is a triangle. Fig. 147, II, represents a beam with breadth constant. The depth varies as the square root of x—the equation of a parabola. One surface may be plane as in II or both may be curved as in Fig. 147, III. In any case the equation gives the total depth. Fig. 147, IV, represents a cantilever in which both depth and breadth vary, all sections being similar rectangles. The equation is that of the cubical parabola.

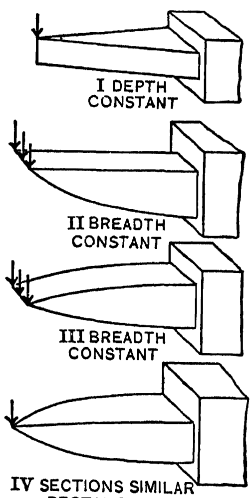


Fig. 147.—Cantilevers of constant strength.

117. Shearing and Bearing Stresses at the End.—In Fig. 147, the load P is represented at the extreme ends of the beams. Allowance must be made at the ends for the bearing and shearing stresses. For instance, in Problem 3 of Article 116, suppose the allowable unit shearing stress to be 150 pounds per square inch. The *average* unit shearing stress in a rectangular section will be 100 pounds per square inch and the minimum area of cross-section will be 6 square inches. The depth at the end should not be less than 1.5 inches.

Suppose also that the allowable bearing stress is 300 pounds per square inch, and that the *center* of the load must be 5 feet from the wall; the bearing area must be at least 2 square inches. If the load extends the entire width of the beam the bearing area must be 4 inches by ½ inch. The actual beam must extend at least ¼ inch beyond the center of the load. Fig. 148 shows the details for these conditions. The dotted lines are the limits for the beam figured for bending only. The solid lines show the *minimum* dimensions figured for all stresses. The actual beam should be

somewhat larger at the end than shown, as a great increase in safety can be secured here with practically no increase in cost and weight. Artistic appearance and convenience of construction may cause further modifications *outside* of the *minimum dimensions*.

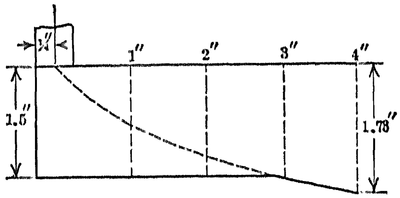


FIG. 148.

Problems

1. Design a cantilever of constant strength for a load of 600 pounds at a distance of 40 inches from a wall: the maximum bending stress to be 800 pounds; the maximum shearing stress, 100 pounds; and the maximum bearing stress, 200 pounds per square inch. The depth of the beam is constant, 4 inches.

2. Design the same cantilever with square section, all other conditions remaining the same as in Problem 1.

118. Cantilever with Uniformly Distributed Load.—The only difference between a cantilever with uniformly distributed load and one with a concentrated load is in the expression for the external moment, which is $\frac{wx^2}{2}$ instead of Px.

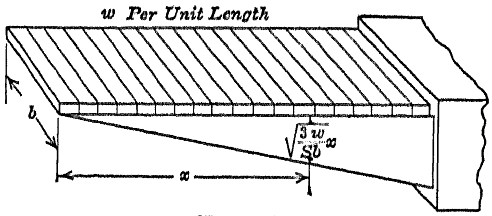


FIG. 149.

Problems

1. A cantilever of constant strength has a rectangular section and constant breadth b. The load is uniformly distributed and is w pounds per inch of length. If S is the allowable unit stress, find the expression for the depth.

Ans. $d^2 = \frac{3\,wx^2}{Sb}$.

2. Draw a cantilever of constant strength and constant breadth of 2 inches to carry a load of 180 pounds per foot uniformly distributed, with an allow-

able unit stress of 1,000 pounds per square inch. The length of the cantilever is 40 inches.

3. A cantilever of constant strength of rectangular section is d inches deep, Fig. 150. If the load is uniformly distributed, find the expression for the breadth.

Ans. $b = \frac{3wx^2}{Sd^2}$.

4. Derive the expression for the depth of a cantilever of square section to carry a uniformly distributed load

Ans. $d^3 = \frac{3wx^2}{S}$.

5. A cantilever of constant strength and square section, designed to carry a uniformly distributed load, is 8 inches square at 50 inches from the free end. Find the dimensions at each 10 inches.

Ans. 2.74, 4.34, 5.69, 6.89, 8.0 inches.

6. Design and draw a cantilever of constant strength and constant breadth of 2 inches to carry a distributed load of 120 pounds per foot and a load of

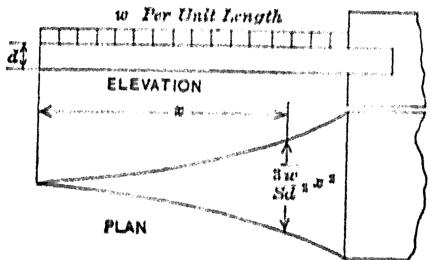


FIG. 150.

400 pounds 6 inches from the free end. The cantilever is 5 feet long; the allowable bending stress is 800 pounds per square inch; the allowable unit shearing stress is 80 pounds per square inch; and the allowable unit bearing stress is 150 pounds per square inch.

119. Beams of Constant Strength Supported at the Ends.—The methods of solution are the same as for cantilevers. For a single load at the middle the problem is exactly the same as that of a cantilever of one-half the length with a load at the end. A beam with a single load at any point may be regarded as made of two cantilevers fixed at that point with loads equal to the respective end reactions. Beams with distributed loads are not quite so simple. Allowance must be made in these beams for shear and bearing at the supports, which was not necessary in the case of cantilevers with uniformly distributed loads.

Problems

1. A cast-steel beam is made for a span of 8 feet to carry a load of 2,400 pounds per foot. Find the section modulus at each foot interval if the allowable bending stress is 12,000 pounds per square inch.

Ans. 8.4, 14.4, 18.0, 19.2 inches.^{3}

2. A box girder is made of two 10-inch 25-pound channels, to which are riveted two 15-inch by ¼-inch plates. The span is 30 feet and the load is 1,800 pounds per foot. The girder is strengthened by additional ¼-inch plates extending equal distances on each side of the middle. If the allowable unit stress is 15,000 pounds per square inch, how many of these plates are required and what is the minimum length of each pair, no allowance being made for weakening of the lower plates due to the rivet holes?

Rolled shapes are not made as beams of "constant strength" but built-up beams, made of shapes and plates are so constructed. In machinery and vehicles, where weight is important, beams of constant strength are much used. In the framework of machinery these are frequently made of cast iron; in other places, cast steel or steel forgings are employed. Cast steel is used extensively in the construction of railway cars, and steel forgings in automobiles.

A tree is a vertical beam of constant strength. A bamboo rod is a hollow beam of constant strength with a large moment of inertia relative to the weight.

120. Deflection of Beams of Constant Strength.—The problem of finding the deflection of a beam of constant strength differs from that of a uniform section in that the moment of inertia is no longer constant but is a function of x. In beams symmetrical with respect to the neutral surface,

$$M = \frac{2SI}{d},$$

where S is constant throughout the length and d may be constant or variable. In the following discussions a constant depth will be represented by capital D and a constant breadth by a capital B, so as to make it easy to distinguish between constants and variables.

121. Deflection of Beams of Constant Depth.—

$$M = EI\frac{d^2y}{dx^2} = \frac{2SI}{D}. \tag{1}$$

DOUBLE INTEGRATION

Dividing (1) by I:

$$E\frac{d^2y}{dx^2} = \frac{2S}{D}. \tag{2}$$

$$E\frac{dy}{dx} = \frac{2Sx}{D} + C_1. \tag{3}$$

If the direction of the X axis be so chosen that $\frac{dy}{dx} = 0$ when

$$x = a, \text{ then } C_1 = -\frac{2\,Sa}{D}.$$

$$Ey = \frac{Sx^2}{D} - \frac{2\,Sax}{D} + C_2. \tag{4}$$

If the origin be chosen so that $y = 0$ when $x = a$, $C_2 = \frac{Sa^2}{D}$,

$$Ey = \frac{S}{D}(x^2 - 2\,ax + a^2) = \frac{S}{D}(a - x)^2. \tag{5}$$

At the origin, $Ey = \frac{Sa^2}{D}$. (6)

AREA MOMENTS

$\frac{M}{I} = \frac{2\,S}{D}$, which is constant and may represent the altitude of a rectangle. Ey = the moment of the $\frac{M}{I}$ diagram. To find the deflection of the origin from the line which is tangent at the distance a from the origin,

$$Ey = \frac{2\,Sa}{D} \times \frac{a}{2} = \frac{Sa^2}{D}. \tag{6}$$

At a distance x from the origin, the base of the rectangle is $a - x$ and its center of gravity is $\frac{a - x}{2}$ from the point whose abscissa is x.

$$Ey = \frac{2\,S(a - x)}{D} \times \frac{a - x}{2} = \frac{S}{D}(a - x)^2. \tag{5}$$

Equations (5) and (6) are valid for beams of constant strength and constant depth no matter what the character of the loading.

For cantilevers $a = l$ and the moment is negative so that

$$Ey_{\max} = -\frac{Sl^2}{D};\ Ey = -\frac{S}{D}(l - x)^2. \tag{7}$$

For a *cantilever with a load on the end*, $S = \frac{PlD}{2\,I_m}$, where I_m is the maximum moment of inertia (at the wall). Substituting,

$$Ey_{\max} = -\frac{Pl^3}{2\,I_m}. \tag{8}$$

The deflection at the end of a cantilever of constant strength and constant depth, due to a load on the end, is one and one-half

times as great as the deflection of a cantilever of uniform section having the same maximum dimensions.

For a *cantilever with uniformly distributed load,* $S = \frac{WlD}{4\,I_m}$, and

$$Ey_{\max} = -\frac{Wl^3}{4\,I_m}. \tag{9}$$

which is twice as great as the deflection of a cantilever of uniform section equal to the maximum dimensions of the constant-strength beam.

For a *beam supported at the ends with a load P in the middle,* $a = \frac{l}{2}$, $S = \frac{PlD}{8\,I_m}$. The deflection of the end upward is

$$Ey_{\max} = \frac{Pl^3}{32\,I_m}, \tag{11}$$

which is one and one-half times as great as that of a beam of uniform section. This might have been taken directly from the result for the cantilever.

For a *beam supported at the ends with a load uniformly distributed*

$$S = \frac{WlD}{16\,I_m} \text{ and } a = \frac{l}{2};$$

$$Ey_{\max} = \frac{Wl^3}{64\,I_m}, \tag{11}$$

which is six-fifths as great as the deflection of a beam of uniform section.

122. Rectangular Beams of Constant Breadth.—

$$\frac{M}{I} = \frac{2\,S}{d}, \text{ where } d \text{ is a function of } x.$$

For a *cantilever with a load on the end,*

$$d^2 = \frac{6\,Px}{SB}, \text{ and } \frac{M}{I} = 2\,S\sqrt{\frac{SB}{6\,P}}\,x^{-\frac{1}{2}}.$$

DOUBLE INTEGRATION

$$E\frac{d^2y}{dx^2} = 2\,S\sqrt{\frac{SB}{6\,P}}\,x^{-\frac{1}{2}}. \tag{1}$$

$$E\frac{dy}{dx} = 4\,S\sqrt{\frac{SB}{6\,P}}\,x^{\frac{1}{2}} + \left[C_1 = -4\,S\sqrt{\frac{SB}{6\,P}}\,l^{\frac{1}{2}}\right]. \tag{2}$$

$$Ey = 4\,S\sqrt{\frac{SB}{6\,P}}\left(\frac{2\,x^{\frac{3}{2}}}{3} - l^{\frac{1}{2}}x + \left[C_2 = \frac{l^{\frac{3}{2}}}{3}\right]\right). \tag{3}$$

$$Ey_{\max} = \frac{4\,Sl^{3/2}}{3}\sqrt{\frac{SB}{6\,P}}. \tag{4}$$

and is negative when P is downward.

At the wall $S = \dfrac{6\,Pl}{BD^2}$, which substituted in (4) gives

$$Ey_{\max} = \frac{2\,Pl^3}{3\,I_m}. \tag{5}$$

The deflection is twice as great as that of a beam of uniform section of which I_m is the moment of inertia.

For a *beam supported at the ends and loaded at the middle,* the deflection at the middle is $\dfrac{Pl^3}{24\,EI_m}$.

For a *cantilever with uniformly distributed load,* $d^2 = \dfrac{3\,wx^2}{SB}$,

$$\frac{M}{I} = \frac{2\,S}{d} = \frac{2\,S}{x}\sqrt{\frac{SB}{3\,w}}.$$

AREA MOMENTS

$$Ey_{\max} = \int \frac{M}{I} x\,dx = 2\,S\sqrt{\frac{SB}{3\,w}}\int_0^l dx = 2\,Sl\sqrt{\frac{SB}{3\,w}} \tag{6}$$

At the wall,

$$S = \frac{3\,wl^2}{BD^2};\ Ey_{\max} = \frac{6\,wl^4}{BD^3} = \frac{Wl^3}{2\,I_m}. \tag{7}$$

which is four times as great as the deflection of a beam of uniform section.

The equation of the elastic line is found (preferably by double integration) to be

$$Ey = -\,2\,S\sqrt{\frac{SB}{3\,w}}\left(x \log \frac{x}{l} - x + l\right). \tag{8}$$

For a *beam supported at the ends with uniformly distributed load, and breadth constant,*

$$M = \frac{w}{2}(lx - x^2);\quad d^2 = \frac{6\,M}{SB} = \frac{3\,w(lx - x^2)}{SB};$$

$$\frac{M}{I} = \frac{2\,S}{d} = 2\,S\sqrt{\frac{SB}{3\,w(lx^2 - x^2)}}. \tag{9}$$

AREA MOMENTS

$$Ey_{\max} = \int \frac{M}{I}\, x\, dx = 2\, S\sqrt{\frac{SB}{3\,w}} \int \frac{xdx}{\sqrt{lx - x^2}}. \tag{10}$$

$$\int \frac{x\, dx}{\sqrt{lx - x^2}} = \frac{x\, dx}{\sqrt{\left(\frac{l}{2}\right)^2 - \left(\frac{l}{2} - x\right)^2}} = -\frac{l}{2}\int \frac{dz}{\sqrt{\left(\frac{l}{2}\right)^2 - z^2}} + \int \frac{zdz}{\sqrt{\left(\frac{l}{2}\right)^2 - z^2}}; \tag{11}$$

where $z = \frac{l}{2} - x;\ dz = -dx.$

$$Ey_{\max} = 2\, S\sqrt{\frac{SB}{3\,w}}\left[-\frac{l}{2}\sin^{-1}\frac{2z}{l} - \sqrt{\left(\frac{l}{2}\right)^2 - z^2}\right]_{z=\frac{l}{2}}^{z=0} \tag{12}$$

(When $x = 0,\ z = \frac{l}{2}$; when $x = \frac{l}{2}$; $z = 0$).

$$Ey_{\max} = Sl\sqrt{\frac{SB}{3\,w}}\left(\frac{\pi}{2} - 1\right). \tag{13}$$

Since $$S = \frac{3\,wl^2}{4\,BD^2},$$

$$Ey_{\max} = \frac{3\,wl^4}{8\,BD^3}\left(\frac{\pi}{2} - 1\right) = \frac{wl^4}{32\,I_m}\left(\frac{\pi}{2} - 1\right) = \frac{6.85\,wl^4}{384\,I_m}. \tag{14}$$

123. Beams of Constant Strength and Similar Sections.—In a beam with square sections, $d^3 = \frac{6\,M}{S}$.

For a *cantilever with load on the end,* $d^3 = \frac{6\,Px}{S}$,

$$\frac{M}{I} = \frac{2\,S}{d} = \frac{2\,S^{\frac{4}{3}}}{(6\,P)^{\frac{1}{3}}\, x^{\frac{1}{3}}}. \tag{1}$$

AREA MOMENTS

$$Ey_{\max} = \frac{2\,S^{\frac{4}{3}}}{(6\,P)^{\frac{1}{3}}}\int_0^l x\, dx = \frac{6\,S^{\frac{4}{3}}\, l^{\frac{5}{3}}}{5\,(6\,P)^{\frac{1}{3}}}. \tag{2}$$

$$S = \frac{6\,Pl}{D^3};\ Ey_{\max} = \frac{36\,Pl^3}{5\,D^3} = \frac{3\,Pl^3}{5\,I_m}. \tag{3}$$

which is nine-fifths as great as that in a beam of uniform section.

The deflection at the middle of a *beam which is supported at the ends* and *loaded at the middle* is given by,

$$Ey_{\max} = \frac{3\,Pl^3}{80\,I_m}. \tag{4}$$

For a *cantilever with uniformly distributed load* the deflection at the end is,

$$Ey_{\max} = \frac{3\,wl^4}{8\,I_m} = \frac{3\,Wl^3}{8\,I_m}. \tag{5}$$

These equations apply to all cases where the sections are similar figures as well as to square sections.

Problem

1. A beam of constant strength and circular section is supported at points 5 feet apart and carries a load of 400 pounds in the middle. The diameter at the middle is 2 inches and E is 30,000,000 pounds per square inch. Find the deflection at the middle.

124. Beams of Two or More Materials.—Beams are frequently made of two or more materials having different moduli of elasticity. The most common cases are beams made by bolting iron or steel plates to wooden beams, and reinforced concrete beams, in which steel rods are embedded in the concrete in the tension side.

Fig. 151 represents a beam made by bolting a steel plate to the

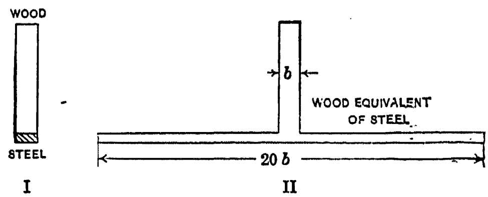


FIG. 151.—Beam of two materials.

bottom of a wooden beam. To find the neutral axis and the section modulus, account must be taken of the fact that the modulus of elasticity of the wood is less than that of the steel, and that with a given deformation, the unit stress in the two materials is proportional to their respective moduli. If the modulus of steel is 30,000,000 and that of the timber is 1,500,000 pounds per square inch, the unit stress in the steel corresponding to a given unit elongation is 20 times as great as the unit stress in the timber.

To find the neutral axis for Fig. 151 the center of gravity may be calculated under the assumption that the density of the steel is 20 times that of the wood, or the steel may be replaced for the purpose of calculation, by a wooden strip, 20 times as wide and of the same thickness.

Example

A 4-inch by 6-inch wooden beam has a steel plate 1 inch wide and ½ inch thick fastened to the lower surface. Find the neutral axis and the maximum fiber stress in each material if the modulus of elasticity of the steel is 20 times as great as that of the wood, and the bending moment is 30,000 inch-pounds.

The steel may be replaced by a wooden strip 20 inches wide and ½ inch thick. To get the distance of the center of gravity from the bottom of the wood,

$$\bar{y} = \frac{24 \times 3 - 10 \times \frac{1}{4}}{34} = 2.04 \text{ inches.}$$

To get the moment of inertia of the equivalent wooden section about the common surface.

$$\frac{4 \times 6^3}{3} = 288,$$

$$\frac{20 \times (\frac{1}{2})^3}{3} = 0.83,$$

$$I = 288.83.$$

$$I_0 = 288.83 - 34 \times 2.04^2 = 147.34 \text{ inches.}^4$$

To get the unit stress in the top fibers of the wood

$$S = \frac{30{,}000 \times 3.96}{147.34} = 806 \text{ pounds per square inch.}$$

In the bottom steel fibers

$$S = \frac{30{,}000 \times 2.54 \times 20}{147.34} = 10{,}344 \text{ pounds per square inch.}$$

The result for steel is multiplied by 20 because the moment of inertia used was calculated on the assumption that the steel was replaced by wood.

Problems

(Use E for steel 20 times E for timber in these problems)

1. A 4-inch by 4-inch timber beam has a 4-inch by ½-inch steel plate on the lower surface and a 2-inch by 1-inch plate on the upper surface. Find the neutral axis of the combination. What is the maximum fiber stress in the steel when that in the wood is 600 pounds per square inch.

Ans. Neutral axis 2.10 inches above bottom of timber; fiber stress in steel, 16,571 pounds per square inch.

2. A 6-inch by 6-inch timber beam, 10 feet long, has a 6-inch by ½-inch steel plate on the top and bottom surfaces. Find the unit stress in the steel when a load of 9,000 pounds is put on the middle.

Ans. 13,720 pounds per square inch.

Fig. 152 is called a flitched beam. It is built up of wooden beams alternating with steel plates, fastened together by means of a few bolts. If the vertical depth of the beams and plates is the same, the bolts do not transmit shear at unit stresses below the elastic limit. With vertical loads, parallel to the plane of the plates, the steel is not used efficiently in beams of this sort.

Flitched beams were once used to some extent, but very little at present. Steel I-beams are usually preferable. Wooden beams are frequently bolted to the web of an I-beam. This is generally done for convenience in attaching woodwork rather than for reinforcing the beam.

When steel is fastened to the top and bottom of a wooden beam, it is used efficiently. The combination is equivalent to an I-beam section, the timber acting as the web and the steel as the flange.

Such combinations are not used in structures, but are employed in vehicles and some classes of machinery. Fig. 153 is a section

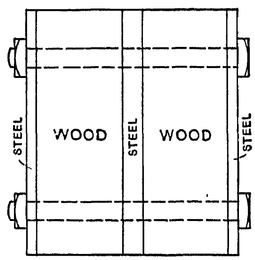


FIG. 152.—Flitched beam.

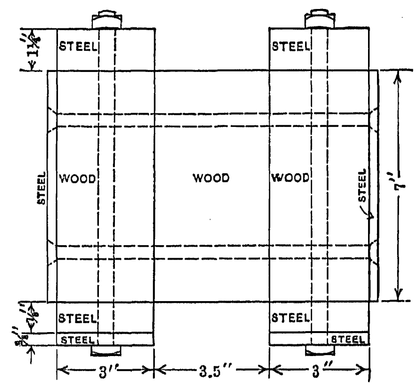


FIG. 153.—Armored wooden beam.

of the lower end of the dipper handle of a steam shovel. It consists of an oak beam made of three pieces (the middle piece is omitted at the upper end) with heavy steel plates at the top and bottom and thin plates on the sides. Where such beams are built up of thin steel shapes the compression side is liable to buckle, especially if it is kinked by rough usage, but with plates fastened to a solid wooden beam, this buckling is prevented. The elastic properties of some wood makes it preferable in many cases where there is considerable vibration.

125. Reinforced-concrete Beams.—Reinforced concrete represents another form of combination beam. A reinforced-concrete beam has steel rods embedded in the concrete near the surface in the tension side. Sometimes both tension and compression sides are reinforced. These rods may be ordinary round or square steel bars. Usually they are corrugated or

otherwise deformed or made of cable or twisted square bar so that they will not slip even if the grip of the concrete should be weakened.

Fig. 154 represents a portion of a reinforced-concrete beam 8 inches by 11 inches in cross-section. The reinforcement consists of three rods with centers 1 inch from the bottom of the beam. The photograph, Fig. 217, shows a beam of this size after failure.

In working out the theory of concrete beams, it is customary to regard the steel as taking all the tension. If the unit stresses

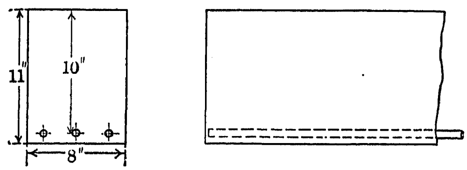


FIG. 154.—Reinforced concrete beam.

are kept low, the concrete on the tension side of the neutral axis does exert some tensile stress, but at loads of less than one-third of the ultimate strength of the beam fine cracks form in the tension side and tests show that the steel takes practically all the tension at larger loads.

The *per cent. of reinforcement* in a beam is calculated by dividing the area of the steel by the area of the beam section above the center of the steel. In Fig. 154 the beam is regarded as an 8-inch by 10-inch section; the inch of concrete below the center of the rods is considered as simply protecting the steel. With three ⅝-inch rods, each of which has a cross-section of 0.307 square inch, the reinforcement in the beam of Fig. 154 is $0.921 \div 80 = 0.0115 = 1.15$ per cent. While it is customary to speak of the per cent. of reinforcement, when used in formulas it is expressed as a ratio.

Elaborate formulas have been proposed for the calculation of reinforced-concrete beams. Some of these formulas assume that the compression curve of concrete is a parabola, *which it is not.* The form and the constants of the compression curve vary greatly with the materials, the proportions, the care in mixing, the age, and the stresses to which it has been subjected. The modulus of elasticity is lowered greatly by slight overloads. For these reasons there is little use for great refinement of calculation unless the computer is provided with carefully determined compression

curves of the actual concrete under consideration, and it is now customary to work on the assumption that the compression curve is a straight line.

A Joint Committee from The American Society of Civil Engineers, The American Society for Testing Materials, The American Railway Engineering Association, and The Association of American Portland Cement Manufacturers has prepared a report on "Concrete and Reinforced Concrete" and has recommended certain formulas and constants. In the articles which follow, the important formulas will be given in the form proposed by the Joint Committee, and with the same symbols except that *s* will be used for unit stress in place of *f*. The constants given in Table XI agree with the report in round numbers.*

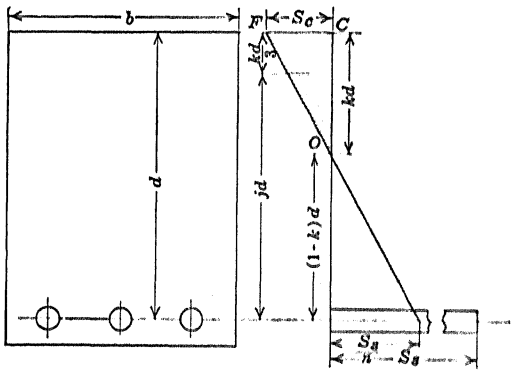


FIG. 155.—Stress in rectangular concrete beam.

126. Location of the Neutral Axis.—The line *OF* of Fig. 155 represents the compressive stress. The depth from the extreme compressive fibers to the center of the reinforcement is *d* and the distance from the neutral surface to the extreme fibers is *kd*, where *k* is a fraction less than unity. The distance from the neutral surface to the center of the reinforcement is $(1 - k)d$. The ratio of the modulus of elasticity of the steel to that of the concrete is represented by *n*;

$$n = \frac{E_s}{E_c}. \tag{1}$$

If the modulus of elasticity of the steel is 30,000,000 and that of the concrete in compression is 2,000,000 pounds per square inch the value of *n* is 15, and this is the value commonly used, al-

* *Proceedings of the American Society for Testing Materials*, vol. XIII, 1913.

though for concrete having an ultimate strength of from 2,200 to 2,900 pounds per square inch, the Joint Committee recommends that $n = 12$, and for concrete stronger than 2,900 pounds per square inch the Committee recommends $n = 10$.

The average unit stress in the steel is given by,

$$s_s = \frac{n(1-k)}{k} S_c, \tag{2}$$

where S_c is the unit compressive stress in the concrete in the extreme fibers. The area of the concrete in compression in a rectangular section is bkd, and the average stress over this area is $\frac{S_c}{2}$.

$$\text{Total compressive stress} = \frac{S_c kbd}{2}. \tag{3}$$

$$\text{Total tensile stress in steel} = \frac{AnS_c(1-k)}{k}, \tag{4}$$

where A is the area of the steel. The ratio of the area to the steel to the area of the concrete is represented by p;

$$p = \frac{A}{bd};\ A = pbd, \tag{5}$$

As the concrete below the neutral surface is not regarded as taking any of the tensile stress, the total tension in the steel equals the total compression in the concrete. Equating (3) and (4) and substituting for A,

$$\frac{S_c kbd}{2} = \frac{S_c pbdn(1-k)}{k}, \tag{6}$$

$$k^2 = 2pn(1-k), \tag{7}$$

$$k^2 + 2pnk - 2pn = 0, \tag{8}$$

$$k = \sqrt{2pn + (pn)^2} - pn. \tag{9}$$

Problems

1. If the modulus of the steel be taken as 15 times that of the concrete and the area of the steel is 1 per cent. of the total area bd, find the distance of the neutral axis from the extreme compression fibers. *Ans.* $k = 0.418\,d$

2. Solve Problem 1 for a reinforcement of 1.2 per cent. and for 1.6 per cent. *Ans.* $0.446\,d$, $0.493\,d$.

127. Relative Unit Stresses in Concrete and Steel.—When the location of the neutral axis has been determined by means of equation (9) of Article 126, or by experiment, the relative values

of the average compressive stress in the concrete and the average tensile stress in the steel may be computed from the relation that the total compression equals the total tension. For instance, in Problem 1 of the preceding article, the area in compression is 0.418 bd and the area of the steel is 0.01 bd so that the average unit stress in the concrete is $\frac{10}{418}$ as great as that in the steel. With a straight-line compression curve for the concrete, the maximum unit stress in the outer fibers of a rectangular section is twice that of the average. If the unit tensile stress in the steel of Problem 1, Article 126, is 12,000 pounds per square inch, the average unit compressive stress in the concrete is 287 pounds per square inch, and the compressive stress in the extreme fibers is 574 pounds per square inch.

The unit compressive stress in the extreme fibers may also be calculated from the relative distances from the neutral surface and the ratio of the two moduli of elasticity. In Problem 1 of Article 126,

$$S_c = \frac{ks_s}{(1 - k)n} \tag{1}$$

TABLE XI.—ALLOWABLE UNIT COMPRESSIVE STRESSES IN EXTREME FIBERS OF CONCRETE BEAMS, IN POUNDS PER SQUARE INCH

Aggregate	1:2:4	1:3:6
Gravel or hard limestone or sandstone..........	650	425
Soft limestone or sandstone..................	500	325
Cinder..................................	200	125

Problems

1. In Problem 2 of the preceding article, calculate the unit stress in the extreme fibers when the average unit tensile stress in the steel is 12,000 pounds per square inch. *Ans.* 645 and 778 pounds per square inch.

2. Solve Problem 1 if the allowable unit stress in the steel is 16,000 pounds per square inch.

With a unit stress in the steel of 12,000 pounds per square inch the unit compressive stress in the concrete with 1.6 per cent. reinforcement is above the allowed value for 1:2:4 concrete for the best material ordinarily used. If the allowable unit stress in the steel is 16,000 pounds per square inch (which is the maxi-

mum recommended by the Joint Committee) even 1 per cent. of reinforcement gives a unit stress in the concrete of over 700 pounds per square inch so that a richer mix than 1:2:4 must be used or the reinforcement kept below 1 per cent. in order to use the steel most efficiently.

Problem

3. If $n = 15$ and $p = 0.008$ what will be the unit stress in the steel when the unit stress in the outer fibers of the concrete, calculated on the assumption that the compression curve is a straight line, is 650 pounds per square inch. *Ans.* 15,600 pounds per square inch.

128. The Resisting Moment.—The resultant compressive stress is at the center of gravity of the triangle *CFO* of Fig. 155. The resultant tensile stress is regarded as being at the center of the reinforcement, so that the arm of the resisting moment is $\left(1 - \frac{k}{3}\right) d$. The term $\left(1 - \frac{k}{3}\right)$ is represented by the single letter j.

$$\text{Resisting moment arm} = \left(1 - \frac{k}{3}\right) d = jd. \qquad (1)$$

Problems

1. What is the resisting moment arm in Problem 1 of Article 126? *Ans.* $jd = 0.86\,d$.

The resisting moment is either total stress multiplied by the moment arm,

$$M = \frac{S_c jkbd^2}{2} = s_s Ajd; \qquad (2)$$

$$S_c = \frac{2\,M}{jkbd^2}; \qquad (3)$$

$$s_s = \frac{M}{Ajd}. \qquad (4)$$

from formula (3), (4) P. 2.?

(In the following problems $n = 15$, $S_c = 650$ lb./in.²)

2. A reinforced-concrete beam for a span of 15 feet is 10 inches wide and 12 inches deep to center of reinforcement. The reinforcement consists of three deformed bars, each having a cross-section of 0.39 square inch. The beam weighs 125 pounds per linear foot. What is the maximum safe load on the middle, based on the compressive strength of the concrete? What is the unit tensile stress in the steel at this load.

Ans. $M = 167{,}000$ inch-pounds; maximum safe load 2,770 pounds.

3. In Problem 2 find the unit stress in the steel by dividing the moment by the resisting arm to get the total tension, and then dividing by the area of the steel. *Ans.* 15,560 pounds per square inch.

4. Design a reinforced-concrete beam for a span of 20 feet to carry a load of 800 pounds per foot including its own weight, using 1 per cent reinforcement.

Moment about the middle of distributed load $= \frac{wl^2}{8}$ foot-pounds

moment about the middle of a [illegible] = Pl

An approximate value of the resisting moment may be computed from the expression:

$$M = 0.8\,d \times As_s.$$

The moment arm is always a little greater than $0.8\,d$ and the total tensile stress in the reinforcement is As_s. Of course, if the percentage of reinforcement is too great the compressive stress in the concrete will be too high.

129. Steel Ratio.—It was shown in Article 127 that when the percentage of reinforcement is too great, the concrete stress will exceed its allowable value before the steel is fully stressed. The ratio of the steel area to total area may be found for any allowable unit stresses. From the equality of the total tensile and compressive stress,

$$\frac{S_c kbd}{2} = s_s A, \qquad (1)$$

from which

$$k = \frac{2\,s_s A}{S_c bd} = \frac{2\,s_s p}{S_c}. \qquad (2)$$

From equation (8) of Article 126,

$$k^2 + 2\,pnk = 2\,pn. \qquad (3)$$

Eliminating k between equations (2) and (3),

$$\frac{4\,s_s^2 p^2}{S_c} + \frac{4\,s_s p^2 n}{S_c} = 2\,pn; \qquad (4)$$

$$p = \frac{n}{2\,\frac{s_s}{S_c}\left(\frac{s_s}{S_c} + n\right)} = \frac{1}{2\,\frac{s_s}{S_c}\left(\frac{s_s}{nS_c} + 1\right)}. \qquad (5)$$

Problem

Find the steel ratio if the allowable unit compressive stress in the concrete is 600 pounds per square inch, the allowable tensile stress in the steel is 15,000 pounds per square inch and the ratio of the modulus of elasticity of the steel to that of the concrete is 15. *Ans.* $p = 0.0075$.

See Report of the Joint Committee on "Concrete and Reinforced Concrete," *Proceedings of The American Society for Testing Materials*, 1913, page 278.

CHAPTER XII

BENDING COMBINED WITH TENSION OR COMPRESSION

130. Transverse and Longitudinal Loading.—It often happens that a beam is subjected to a direct tension or compression in the direction of its length and a transverse force producing a bending moment. The unit stress at any point in a given section is the sum of the direct stress and the bending stress at that point. For example, suppose a 4-inch by 4-inch post stands vertical and supports a load of 4,000 pounds at the top. The direct compressive stress is 250 pounds per square inch. Suppose this post is fixed at the bottom (Fig. 156), and that a horizontal push of 200 pounds is applied 2 feet from the bottom. This transverse force produces a tensile stress of 450 pounds per square inch in the outer fibers at the bottom on the side of the push and a compressive stress of the same magnitude in the opposite side. The resultant stress is 700 pounds per square inch in the one side and 200 pounds per square inch in the other. Fig. 156, IV, shows the distribution of the stress, compression being represented by the vertical distance downward. In Fig. 156, II, there is the compression alone due to the direct load of 4,000 pounds. Fig. 156, III, shows the unit stress due to bending; it is 450 pounds per square inch compression on the left side and 450 pounds per square inch tension on the right. At the middle of the section it is zero. In Fig. 156, IV, the two unit stresses are combined. The line EF, which is the zero line for

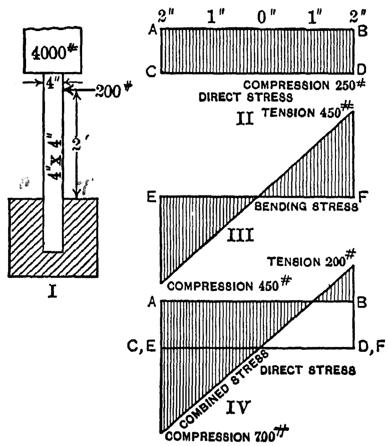


FIG. 156.—Post with compression and bending.

the bending stress, is placed on the line CD, which represents the compressive stress in II. The combined unit compressive stress on the left side is $250 + 450 = 700$ pounds per square inch. On the right side the combined stress is 450 tension minus 250 compression, or 200 pounds per square inch tension. The unit stress is zero at $\frac{8}{9}$ inch from the right side of the post.

$$\text{Unit stress} = \frac{P}{A} + \frac{Mv}{I}, \qquad \text{Formula XXIV.}$$

where P is the total load parallel to the length of the beam and M is the bending moment from any source whatever. Since v has the positive sign on one side of the neutral axis and the negative sign on the other side, the second term may be positive or negative, according to the position.

Problems

1. A wooden post 6 inches square and 6 feet high carries a load 7,200 pounds on the top and is pushed south by a horizontal force of 240 pounds applied 1 foot from the top. Find the maximum unit tensile and compressive stress at the bottom.

Ans. 600 pounds per square inch compressive stress on south.
200 pounds per square inch tensile stress on north.

2. At what distance from the top is the unit tensile stress in north face equal to zero? *Ans.* 42 inches.

3. A solid concrete wall 12 feet high and 6 feet thick is subjected to a water pressure which varies as the depth and which is 62.5 pounds per square foot at a depth of 1 foot. Find the unit stress at the bottom of the wall on each side when the water just reaches the top, the weight of a cubic foot of concrete being 150 pounds.

Ans. 8.33 pounds per square inch tension.
33.3 pounds per square inch compression.

4. What load per running foot on the wall of Problem 3 will reduce the unit tensile stress to zero?

131. Eccentric Loading.—Fig. 157 represents a rigid bar G supported by three equal rubber bands (or springs) which are symmetrically placed and suspended from a rigid horizontal support. Each of the bands is stretched the same amount and the bar hangs in a horizontal position. Fig. 157, II, shows the same bar with a load P at the middle. The rubber bands are equally stretched and the bar remains in a horizontal position. If the load P be moved to the right, as in Fig. 157, III, the middle band receives the same elongation as in the preceding case, while the left band is elongated less and the right band more. If the

load be moved still farther to the right, a place is reached where the left end is elevated above the position which it occupied before the load was applied, so that no load whatever comes on the left band. If, instead of the rubber bands, helical springs were used, the spring on the side away from the load would come into compression when the load is shifted a considerable distance from the middle.

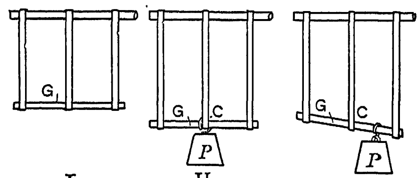


FIG. 157.—Eccentric loading of rubber bands.

Instead of the rubber bands a continuous sheet of rubber or metal might be used. If such a sheet is fastened to a rigid body at the top and bottom and a load is applied considerably to one side of the center, there is elongation on that side and a shortening or buckling on the other. A similar result obtains when a compressive load is applied to a body. Fig. 158 shows a block of soft rubber with the load central, and Fig. 158, II, shows the effect of moving this load a little to one side.

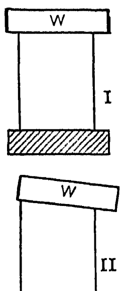


FIG. 158. —Compression with direct and with eccentric loading.

Considering the bar *G* of Fig. 157, III, it is seen that the effect of the eccentric load is a translation downward, of the same magnitude as that due to the central load in II, together with a rotation about the bottom of the middle band as an axis. Taking moments about the point *C*, equilibrium occurs when the moment of the load *P* with respect to *C* is equal to the moment of the excess of tension on the right band plus the moment of the deficiency of tension on the left band. Suppose that the bands are 4 inches apart, and that a load of 1 pound stretches the bands 0.4 inch. One pound will stretch a single band 1.2 inches. Now move the load of 1 pound 2 inches to the right of the middle. The moment of the load is 2 inch-pounds, which is balanced by a force of 0.5 pound at 4 inches. The tension in the right band is 0.25 pound more than that in the middle, and the tension in the left band is 0.25 pound less. If the load is moved more than 3 inches from the middle, the tension in the left band becomes less than it was before the load was applied.

Fig. 159, I, shows a short block B resting on a rigid support and carrying a rigid block C. Fig. 159, II, shows the plan and elevation of the block B compressed by a force transmitted by the block C. Block C is not shown in Fig. 159, II. It is assumed that C is perfectly rigid, or is so loaded that its lower surface and the upper surface of B remain plane. The resultant of all the forces transmitted through the upper block is P, and, for the present, it is assumed that the line of action of this resultant

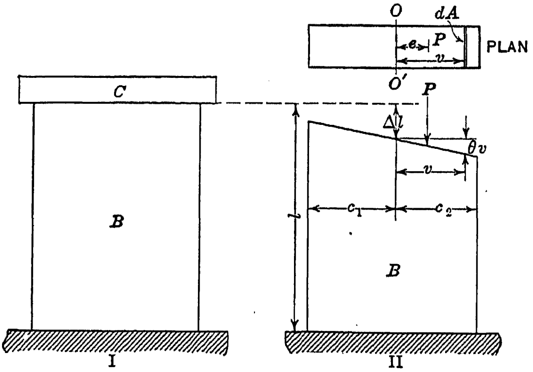


FIG. 159.—Short block with eccentric loading.

passes through one of the principal axes of inertia of the section at a distance e from the center of gravity. The line 0–0′ in the plan is the other principal axis of inertia. The distance e is called the eccentricity of the load. The base of B is supposed to remain horizontal while the upper surface makes an angle θ with the horizontal. If Δl is the compression of the fibers at the center of gravity, at a distance v from 0–0′, the total compression is $\Delta l + v\theta$.

$$\text{Unit compression} = \frac{\Delta l + v\theta}{l}. \tag{1}$$

$$\text{Unit stress} = \frac{E(\Delta l + v\theta)}{l}. \tag{2}$$

On an element of area dA at a distance v from the center of gravity of the section

$$\text{stress} = \frac{E(\Delta l + v\theta)}{l}\,dA. \tag{3}$$

The total stress on the entire section is

$$P = \int_{c_1}^{c_2} \frac{E\Delta l}{l}\,dA + \int_{c_1}^{c_2} \frac{Ev\theta}{l}\,dA. \tag{4}$$

Since e is measured from a line through the center of gravity the last integral of (4) is zero and

$$P = \int_{c_1}^{c_2} \frac{E\Delta l}{l} dA = \frac{E\Delta l}{l} A = s_c A. \tag{5}$$

where s_c is the unit compressive stress at the center of gravity of the section.

The unit deformation and unit stress at the center of gravity are the same with an eccentric load as they would be if the load were central.

Taking moments about the line 0–0′ or any parallel line through the center of any section,

$$Pe = \frac{E\Delta l}{l}\int v dA + \frac{E\theta}{l}\int v^2 dA \tag{6}$$

The first integral of (6) vanishes so that,

$$Pe = \frac{E\theta}{l}\int v^2 dA = \frac{E\theta I}{l}; \tag{7}$$

$$\frac{Pev}{I} = \frac{E\,\theta\, v}{l},$$

which is the unit stress at a distance v from the center of gravity caused by the additional elongation due to the rotation of the top surface about 0–0′. The entire unit stress at a distance v from the line 0–0′ is,

$$s_a = \frac{P}{A} \pm \frac{Pev}{I} = \frac{P}{A} \pm \frac{Mv}{I}, \quad \text{Formula XXIV.}$$

where M is the moment of the resultant force about a line through the center of gravity of the section. When e and v are on the same side of the center of gravity the sign of the last term is positive. When they are on opposite sides the sign is negative.

Formula XXIV as applied to eccentric loading, might be derived from the principle of Mechanics* that a force along any line may be replaced by an equal force along any other line, and a couple whose moment is the product of either force multiplied by the distance between them. In Fig. 160 the load P at the top at a distance e from the center may be replaced by force P at the center and a couple the moment of which is Pe and the direction of rotation is clockwise. The two equal and oppositely directed forces which comprise this couple may be regarded as

* HOSKINS' "Theoretical Mechanics," Art. 94; MAURER'S "Technical Mechanics," Art. 31.

having any magnitude, position, and direction in their plane, provided the product of one force by the distance between them is Pe. The reaction at the bottom may likewise be regarded as equivalent to a reaction P at the center and a counterclockwise moment Pe. An eccentric load may be regarded as equivalent to

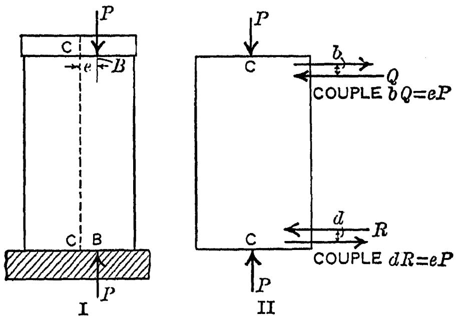


FIG. 160.—Block with eccentric loading.

a load at the center and a bending moment which is the product of the load multiplied by its eccentricity.

Fig. 161 shows large eccentricity, the resultant load lying entirely outside of the section. Here the existence of bending and direct stress together is almost self-evident. Consider the por-

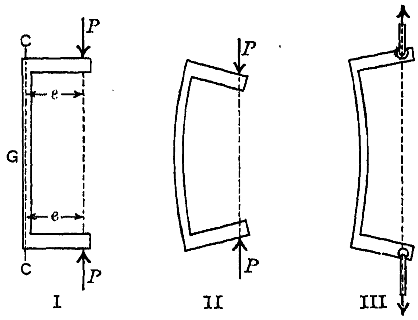


FIG. 161.—Large eccentricity.

tion above any section G. Resolving vertically, the vertical reaction at the section is equal to the load P at the top. Taking moments about an axis perpendicular to the plane of the paper through the center of the section G, the resisting moment of the section must equal the moment eP. Fig. 161, II, shows the effect of compression and Fig. 161, III, the effect of tension.

Formula XXIV as applied to eccentric loading assumes that the section at which the load is applied remains plane. If the load is concentrated, this will not be the case and the results of the formula will be only approximate. If the block under stress is of some length, the sections near the middle are practically plane and the formula applies with greater accuracy.

The derivation also assumes that E is constant, which limits the formula to unit stresses below the elastic limit. Since the

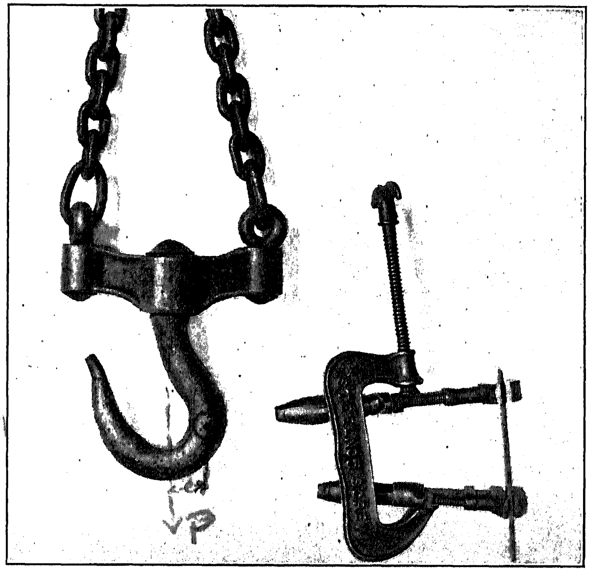


FIG. 162.—Eccentric loading.

stress on one side is greater than on the other, it will reach the elastic limit before the other and there will be a shifting of the neutral axis, so that the error due to this cause will be greater than in a beam subjected to bending alone.

Formula XXIV applies approximately to hooks, subject to the correction for curvature. In any case, when there is appreciable deflection, the eccentricity is measured from the center of gravity of the loaded section to the line of the load.

Compression combined with bending is shown in Fig. 162. The forces P are applied to the wrenches by the screw clamp. The wrenches as cantilevers transmit the bending moments and direct compression to the bar. The experiment may easily be

performed by two wrenches and a steel or wooden bar, the force being applied to the wrenches by hand. The bar will bend as in Fig. 161, II, if the forces are toward each other, and as in Fig. 161, III, if the forces are from each other.

The clamp of Fig. 162 is subjected to *tension* and bending. The eccentricity is the distance from the center of the screw to the center of gravity of any section. In a hook the load line joins the shank with the point which is immediately below it when loaded. This point is, of course, the point in the concave portion which is farthest from the shank. The eccentricity is the distance of this load line from the center of gravity of the section.

Problems

1. A 4-inch by 6-inch short block is subjected to a compressive load of 6,000 pounds, the line of the resultant load being ½ inch from the axis of the block, and in the plane parallel to the 6-inch faces, which passes through the axis. Find the maximum compressive stress and the minimum stress.

Ans. Maximum, 375 pounds per square inch.
Minimum, 125 pounds per square inch compression.

2. In Problem 1 solve for the case when the load is 1 inch from the axis.
Ans. 500 pounds per square inch, 0.

3. A solid circular rod 2 inches in diameter is subjected to a pull of 15,000 pounds. Find the maximum and minimum unit stress if the line of the load is 0.1 inch from the axis of the rod.

Ans. 6,684 and 2,865 pounds per square inch tension.

4. Solve Problem 3 if the line of the load is 0.3 inch from the axis of the rod. *Ans.* 10,505 pounds per square inch tension.
955 pounds per square inch compression.

5. Solve Problem 4 if the rod is hollow with inside diameter 1 inch.
Ans. 12,478 pounds per square inch tension.
254 pounds per square inch tension.

6. A block b wide and d thick, of rectangular section, has the load so placed that the unit stress in the outer fibers on one side is zero. If the line of load is in the plane of symmetry parallel to the faces of breadth d, what is the eccentricity? *Ans.* $\frac{d}{6}$.

7. What eccentricity in a solid circular section of radius a will make the unit stress on one side zero? *Ans.* $e = \frac{a}{4}$.

8. A hollow circular cylinder of outside radius a and inside radius b is so loaded that the unit stress on one side is zero. What is the eccentricity?
Ans. $e = \frac{a^2 + b^2}{4a}$.

9. A solid wall has the resultant load 2 feet from the front edge. The load is 12 tons per running foot. Assuming that the load is so distributed that the top remains plane, find the unit stress in tons per square foot at the front edge if the breadth of the wall is 4 feet, 6 feet, 8 feet, 10 feet.

Ans. 3, 4, 3.75, 3.36 tons per square foot compression.

10. In a hook of circular section the distance from the center of gravity of the section to the line of the load is 3 inches. The load is 1,600 pounds and the diameter of the section is 2 inches. Using Formula XXIV, find the *approximate* value of the maximum tensile and compressive stress.

Ans. 6,621 pounds per square inch tension.
5,602 pounds per square inch compression.

132. Relation of Maximum Unit Stresses to Form of Section.—Formula XXIV may be written,

$$s = \frac{P}{A}\left(1 + \frac{ev}{r^2}\right),$$

where r is the radius of gyration of the section with respect to an axis through the center of gravity. Let c_1 be the distance to the extreme fiber on the side of the eccentric load, and let c_2 be the distance to the extreme fiber on the opposite side. If it is desired that the unit tensile stress and the unit compressive stress at the outer fibers shall be numerically equal,

$$1 + \frac{ec_1}{r^2} = -\left(1 - \frac{ec_2}{r^2}\right) = \frac{ec_2}{r^2} - 1; \tag{1}$$

$$c_2 - c_1 = \frac{2\,r^2}{e}. \tag{2}$$

With a given section, so that c_1, c_2, and r are known, equation (2) makes it possible to find at once the eccentricity at which the two units stresses at the outer fibers are numerically equal.

If it is desired that one unit stress shall be n times as great as the other,

$$n + \frac{nec_1}{r^2} = \frac{ec_2}{r^2} - 1; \tag{3}$$

$$c_2 - nc_1 = \frac{(n+1)r^2}{e}. \tag{4}$$

Problems

1. In a bar of circular section subjected to tension, the eccentricity is one-half the radius. What is the ratio of the maximum unit compressive stress to the maximum unit tensile stress? *Ans.* 1/8.

2. Solve Problem 1 if the eccentricity is one-half the radius.

3. A short piece of a 10-inch 15-pound channel in compression has the resultant load 0.4 inch from the center of gravity of the section on a line through the center of gravity perpendicular to the web on the side of the web on which the flanges are located. Find the ratio of the maximum unit compressive to the maximum unit tensile stress.

133. Maximum Eccentricity without Reversing Stress.—A brick pier laid in lime mortar has practically no tensile strength, and the tensile strength of masonry laid in cement is not reliable. For this reason the load on a masonry pier should always be placed so that the stress over the *entire section* shall be *compressive.* Problem 6 of Article 131 showed that a load on a rectangular section at a distance from the center greater than one-sixth of the dimension of the section in this direction produces a negative stress in the outer fibers in the opposite side. For this reason it is a rule of architects and engineers that the *resultant load* shall *not fall outside* of the *middle third* in the case of rectangular columns or piers.

The statement that the load must lie inside the middle third means that if the load is on the line BD, Fig. 163, through the center of the rectangle parallel to the side d, it must lie in the middle third of this line. In the same way, if the load falls on the line FG, it must be between the points C_1C_1 on this line. CC is one-third of the length BD.

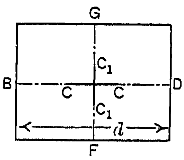


FIG. 163.—Maximum eccentricity on principal axes of rectangular section.

From Problem 7 of Article 131 it is seen that with piers of solid circular section, the load must not lie outside of a circle whose diameter is one-fourth that of the pier. With a hollow pier the circle may be greater without reversing the stress, so that with eccentric loading in the case of materials which are weak in tension, a hollow pier may be stronger than a solid one of the same outside diameter.

For other sections the maximum eccentricity without reversing stress may easily be calculated from Formula XXIV. Using the negative sign, since the fibers with zero stress are on the side of the center of gravity opposite to the eccentricity,

$$s = 0 = \frac{P}{A} - \frac{Pec}{I} = \frac{P}{A}\left(1 - \frac{ec}{r^2}\right); \qquad (1)$$

See previously equation

$$e = \frac{r^2}{c}. \qquad (2)$$

Problems

1. A hollow pier is 24 inches square on the outside and 16 inches square inside. How great may be the eccentricity on a line through the center parallel to the sides without reversing the stress?

Ans. $e = \dfrac{832}{12 \times 12} = 5\frac{7}{9}$ inches.

$r = \sqrt{\frac{I}{A}}$ $\quad \frac{r^2}{c}$

2. In a solid pier 12 inches square, how far may the resultant be placed from the center of gravity of the section if it is on a line through the center parallel to two faces? *Ans.* 2 inches.

3. Solve Problem 2 if the load is on a diagonal. *Ans.* 1.41 inches.

4. A square section of side b has the resultant load at a point C, the coördinates of which are (x, y), Fig. 164, I. Show that when the unit stress at F is zero, the position of C satisfies the equation

$$6\,x + 6\,y = b.$$

SUGGESTION.—The moment of inertia of a square section being the same for all axes through the center, the rotation will be about the axis OE perpendicular to OC. The distance of the extreme fibers at F from this axis is equal to EB.

The distance

$$EB = \frac{b}{2}(\cos\theta + \sin\theta).$$

$$I = \frac{b^4}{12}.$$

For zero stress at the corner, F,

$$e(\cos\theta + \sin\theta) = \frac{b}{6}.$$

$$x + y = \frac{b}{6}.$$

134. Resultant Load not on a Principal Axis.—In all the problems of the preceding articles, the resultant load fell on one principal axis and rotation took place about the other principal

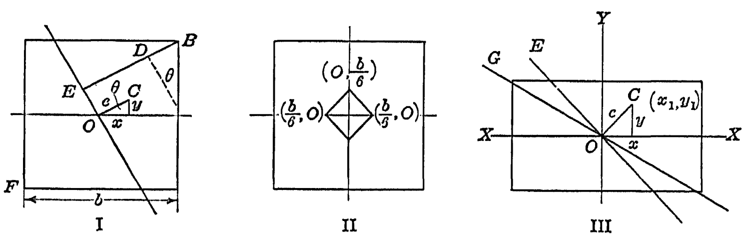


FIG. 164.—Eccentric load not on principal axis.

axis. In the case of a round or square section, the moment of inertia is the same for all axes through the center of gravity, and any such axis may be regarded as a principal axis. In other sections, when the load does not fall on a principal axis, the axis of rotation is not the line OE normal to OC, but is some line OG (Fig. 164, III) between OE and the axis for which I is a minimum.

As in the case of beams where the bending moment is not in the plane of a principal axis of inertia (Article 68), it is necessary to resolve the moment into two components perpendicular to these axes and calculate separately the unit stress at any point due to these components. The actual stress at the point is the sum of

these and the direct stress. In Fig. 164, III, X–X and Y–Y are the two principal axes of inertia, and the resultant load on the section passes through the point C, the coördinates of which with respect to the principal axes are x and y.

The component of the moment about the X axis is Py, and about the Y axis is Px. The bending stress at a point (x_1, y_1) due to the moment Px is $\frac{Pxx_1}{I_y}$, where I_y is the moment of inertia with respect to the Y axis which is perpendicular to the plane of the moment.

The total unit stress at any point (x_1y_1) is,

$$s = \frac{P}{A} + \frac{Pxx_1}{I_y} + \frac{Pyy_1}{I_x} \tag{1}$$

If x and x_1 have the same sign, the second term of the second member of (1) is positive, and if y and y_1 have the same sign the third term is positive.

Example

A rectangular block 12 inches long, measured from east to west, and 10 inches wide is subjected to a load of 3,600 pounds 2 inches from the east edge and 2 inches from the north edge. Find the unit stress at each corner.

The bending stress at the north and south edges due to the couple of 10,800 inch-pounds is 54 pounds per square inch. The bending stress at the east and west edges due to the couple of 14,400 inch-pounds is 60 pounds per square inch. The direct compression is 30 pounds per square inch. The unit stresses at the corners are: 144 pounds compression at the northeast corner; 24 pounds compression at the northwest corner; 84 pounds tension at the southwest corner; 36 pounds compression at the southeast corner.

Problems

1. A rectangle 8 inches long and 6 inches wide has a load of 1,200 pounds applied $2\frac{1}{2}$ inches from one 8-inch side, and $3\frac{1}{3}$ inches from one 6-inch side. Find the unit stress at each corner.

Ans. 50, 25, 0, 25 pounds per square inch.

2. A rectangular post 6 inches by 10 inches has the 10-inch sides in the meridian. A load of 2,400 pounds is applied 1 inch north and 1 inch east of the middle. A horizontal push of 200 pounds is 20 inches above the bottom directed south 30 degrees west, the line of the push passing through the center of the section. Find the unit stress at each corner at the bottom.

Ans.
- 36.03 pounds at northeast corner.
- 22.69 pounds at northwest corner.
- 43.97 pounds at southwest corner.
- 57.31 pounds at southeast corner.

To find the maximum eccentricity in any direction without reversing stress.

$$0 = \frac{P}{A}\left(1 \pm \frac{xx_1}{r_y^2} \pm \frac{yy_1}{r_x^2}\right); \qquad (2)$$

where r_y is the radius of gyration with respect to the Y axis. It will usually happen that both the second and third term will be negative.

Example

Find the maximum eccentricity of the load in the case of a rectangular block of breadth b and thickness d without reversing the stress at the corners.

$$x_1 = \frac{b}{2};\ y_1 = \frac{d}{2};\ r_x^2 = \frac{d^2}{12};\ r_y^2 = \frac{b^2}{12}.$$

The maximum bending stress of sign opposite to the direct stress will be at the corner in the third quadrant if the load is in the first quadrant, and both the second and the third terms of (2) will be negative. Equation (2) becomes

$$1 - \frac{6\,x}{b} - \frac{6\,y}{d} = 0. \qquad (3)$$

This is the equation of a straight line, the intercepts of which are

$$x = \frac{b}{6},\ y = \frac{d}{6}.$$

CHAPTER XIII

COLUMNS

135. Definition.—In the discussion of the preceding chapter, no consideration was made of the deflection of the body and the effect of this deflection in changing the amount of eccentricity. In tension the deflection is such as to diminish the eccentricity (Fig. 161, III). In compression, on the other hand, the deflection increases the eccentricity and consequently increases the unit stress (Fig. 161, II). A yard stick may be placed with one end on the floor and a compressive force applied with the hand to the other end. When the force reaches a certain amount, the stick suddenly bends and may deflect several inches from the straight line. The original eccentricity of possibly 0.01 inch is increased to several inches and the unit stress may be sufficient to cause rupture. If the stick is placed with one end on a platform scale, as in Fig. 165, it is found that the load which produces a deflection of 2 inches is little, if any, greater than the load which causes a deflection of 1 inch. The resisting moment has been nearly doubled, but the external moment has likewise been doubled, owing to the increased length of the moment arm.

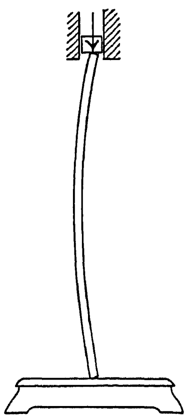


FIG. 165.—A long column.

A compression member whose length is several times as great as its smallest transverse dimension is called a *column*. There is no definite ratio of length to diameter at which a compression member ceases to be a short block and becomes a column. The formulas of Article 137 will show, that when the ratio of the length to the smallest transverse dimension is less than 10, the error made by neglecting the deflection is so small that it may ordinarily be neglected. Some engineers call a compression member of length less than 15 diameters a *short block* or pier and calculate it by the methods of the preceding chapter.

Columns may be vertical, as the intermediate posts of bridges, or horizontal, as the top chords of a bridge. The connecting rod of an engine is a column during the forward stroke. When a column is vertical, the only bending moment is that due to the eccentricity of the load and the deflection. When a column is horizontal or inclined, its own weight applied as a beam becomes an appreciable factor. The rafters supporting a roof act as columns and inclined beams.

A compression member of some length is frequently called a *strut.*

136. Column Theory.—Fig. 166 represents a vertical column with ends free to turn without friction about horizontal axes perpendicular to the plane of the paper. Fig. 166, I, represents the actual column with deflection somewhat exaggerated, and Fig. 166, II, shows the central axis of the column with all horizontal distances enlarged. In order that Formula XV may apply without change of letters, the X axis is taken vertical (parallel to the length of the column) and the Y axis is horizontal and positive toward the left (so that in turning from the X axis to the Y axis the rotation is counterclockwise). The column might be taken horizontal and the bending moment neglected with the same result. The origin of coördinates is at the lower end at the center of gravity of the section. At a section at a distance x from the origin, the moment arm with respect to an axis through the center of gravity of the section normal to the plane of the load and the axis of the column is $y + e$, and

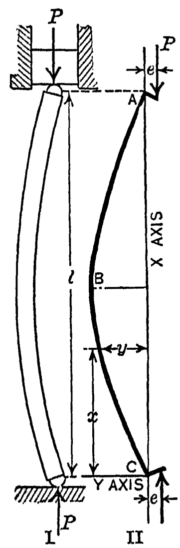


FIG. 166.—Column deflection.

$$\text{Moment} = P(y + e).$$

The eccentricity e is measured from the line of the load to the center of gravity of the section in order that its sign may be the same as the sign of y. When the load is on the negative side of the X axis (as in Fig. 166), so that e and y are positive, the center of curvature is on the negative side and, consequently, the second derivative, $\frac{d^2y}{dx^2}$, is negative. In the same way, if the eccentricity were on the other side of the column so that the deflection would

come on the right (negative), $\frac{d^2y}{dx^2}$ would always be positive. The second derivative has the negative sign when y and e are positive and the positive sign when y and e are negative.

Since the signs of the second derivative and the moment arm are opposite, the differential equation is

$$EI\frac{d^2y}{dx^2} = -P(y + e). \tag{1}$$

This equation may be written

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = -\frac{eP}{EI}, \tag{2}$$

which is a differential equation of the second order and first degree with the right-hand member constant. The student familiar with Differential Equations will write it:

$$\left(D^2 + \frac{P}{EI}\right)y = -\frac{eP}{EI}. \tag{3}$$

The solution of (3) is:

$$y = C_1 \cos\sqrt{\frac{P}{EI}}\,x + C_2 \sin\sqrt{\frac{P}{EI}}\,x - e. \tag{4}$$

The term $\sqrt{\frac{P}{EI}}\,x$ is an angle in radians. (Notice that x is not under the radical sign. The angle may be written $\sqrt{\frac{Px^2}{EI}}$. It does not apply to any angle on the diagram.)

The student who is not familiar with Differential Equations may *verify* (4) by performing the inverse operations to get (2). A solution of a differential equation may always be proved by differentiating and eliminating the constants of integration. Since this equation is of the second order, two differentiations are necessary. Differentiate (4) twice to get $\frac{d^2y}{dx^2}$ and substitute this and the value of y from (4) in (2). The result is an identity, which proves that (4) is a solution of (2).

To obtain the integration constants, the conditions are

$$y = 0 \text{ when } x = 0, \text{ and } y = 0 \text{ when } x = l.$$

From the first condition:

$$C_1 \cos 0 - e = 0;$$

$$C_1 = e; \tag{5}$$

$$y = e\cos\sqrt{\frac{P}{EI}}\,x + C_2 \sin\sqrt{\frac{P}{EI}}\,x - e. \tag{6}$$

Substituting the second condition in (6):

$$0 = e \cos \sqrt{\frac{P}{EI}}\, l + C_2 \sin \sqrt{\frac{P}{EI}}\, l - e; \tag{7}$$

$$C_2 = \frac{e\left(1 - \cos \sqrt{\frac{P}{EI}}\, l\right)}{\sin \sqrt{\frac{P}{EI}}\, l} = \frac{e \sin^2 \sqrt{\frac{P}{EI}}\, \frac{l}{2}}{\sin \sqrt{\frac{P}{EI}}\, \frac{l}{2} \cos \sqrt{\frac{P}{EI}}\, \frac{l}{2}}$$

$$= e \tan \sqrt{\frac{P}{EI}}\, \frac{l}{2}; \tag{8}$$

$$y = e\left(\cos \sqrt{\frac{P}{EI}}\, x + \tan \sqrt{\frac{P}{EI}}\, \frac{l}{2} \sin \sqrt{\frac{P}{EI}}\, x - 1\right). \tag{9}$$

Equation (9) gives the deflection at any point of a column with ends free to turn but not free to move laterally. It is a sine curve. To find the point of maximum deflection, differentiate and set the first derivative equal to zero and find that $x = \frac{l}{2}$ is the position required. We might have assumed, from the symmetry of the figure, that the maximum deflection is at the middle and used this instead of the second condition in getting the constant C_2.

To get the maximum deflection at the middle:

$$y_{\max} = e\left(\cos \sqrt{\frac{P}{EI}}\, \frac{l}{2} + \tan \sqrt{\frac{P}{EI}}\, \frac{l}{2} \sin \sqrt{\frac{P}{EI}}\, \frac{l}{2} - 1\right);$$

$$y_{\max} = e \sec \sqrt{\frac{P}{EI}}\, \frac{l}{2}\left(\cos^2 \sqrt{\frac{P}{EI}}\, \frac{l}{2} + \sin^2 \sqrt{\frac{P}{EI}}\, \frac{l}{2}\right) - e;$$

$$y_{\max} = e\left(\sec \sqrt{\frac{P}{EI}}\, \frac{l}{2} - 1\right); \tag{10}$$

$$y_{\max} + e = e \sec \sqrt{\frac{P}{EI}}\, \frac{l}{2}. = e \sec \sqrt{\frac{Pl^2}{4\,EI}}. \quad \text{Formula XXV.}$$

$$\text{Maximum moment} = eP \sec \sqrt{\frac{Pl^2}{4\,EI}}. \tag{12}$$

$$\text{Maximum unit stress} = \frac{P}{A} + \frac{ePc}{I} \sec \sqrt{\frac{Pl^2}{4\,EI}}; \tag{13}$$

$$\text{Maximum unit stress} = \frac{P}{A}\left(1 + \frac{ec}{r^2} \sec \sqrt{\frac{Pl^2}{4\,EI}}\right); \tag{14}$$

$$\text{Maximum unit stress} = \frac{P}{A}\left(1 + \frac{ec}{r^2} \sec \sqrt{\frac{P}{AE}}\, \frac{l}{2\,r}\right). \tag{15}$$

The ratio of the length of a column to its least radius of gyration $\left(\frac{l}{r}\right)$ is called its *slenderness ratio*.

Slenderness Ratio

Problems

1. A wooden bar 1 inch square and 5 feet long, as a column with round ends, has the line of the load 0.1 inch from the center of the sections. When the load is 200 pounds and E is 1,500,000 pounds per square inch, what is the deflection at the middle?

Ans. $\sqrt{\dfrac{Pl^2}{4\,EI}} = 1.2$ radians; $y_{\max} + e = 0.276$ inch; $y_{\max} = 0.176$ inch.

2. In Problem 1 find the maximum moment and the maximum compressive stress if the load is on a line through the center of the section parallel to one side.

Ans. 55.18 inch-pounds; 531 pounds per square inch.

3. Solve Problem 2 if the load is on the diagonal of the section.

Ans. 668 pounds per square inch.

4. A 2-inch round steel rod 10 feet long is used as a column with ends free to turn. Find the deflection at the middle and the maximum fiber stress on the concave side when the load is 8,000 pounds and the eccentricity is 0.1 inch, if E is 30,000,000 pounds per square inch.

Ans. $y_{\max} + e = 0.1 \sec 63° \, 21' = 0.2230$ inch.
Maximum $S_c = 4{,}814$ pounds per square inch.

5. What would be the deflection and maximum unit stress in Problem 4 if the load were made 10,000 pounds?

6. A column with ends free to turn is made of a 2-inch round steel rod for which E is 30,000,000 pounds per square inch. The length is 5 feet. Find the deflection at the middle and the maximum unit stress for loads of 20,000 pounds, 30,000 pounds, 50,000 pounds, 60,000 pounds, and 70,000 pounds for eccentricities of 0.01 inch and 0.1 inch.

Ans. Load		20,000,	30,000,	50,000,	60,000,	70,000.
Unit stress	For $e = 0.01$,	6,763	10,340	19,295	32,483,	infinite.
	For $e = 0.1$,	10,337	17,500	49,800	154,000,	infinite.

Consideration of the answers of Problem 6 shows that the unit stress for a load of 50,000 pounds with an eccentricity of 0.01 inch is only a little greater than the unit stress for a load of 30,000 pounds with an eccentricity of 0.1 inch. For the load of 50,000 pounds and eccentricity of 0.1 inch the calculated unit stress is 49,800 pounds per square inch. With ordinary steel this would produce failure. A load of 64,570 pounds will cause failure with a column of these dimensions, no matter how small the eccentricity, for this load makes the angle $\sqrt{\dfrac{Pl^2}{4\,EI}}$ equal to $\dfrac{\pi}{2}$, the secant of which is infinity. If it were possible to have the load exactly central, when the load reaches this critical value the column would be in a state of unstable equilibrium; the least vibration would start it to bending, and it would continue to bend without increase of load until it failed.

The formulas of this article are calculated on the assumption that E is constant. They are valid, therefore, only to the proportional elastic limit. While tests have shown that columns made of one piece will support loads considerably above the yield point of the material, tests of built-up columns show that these fail when the yield point is reached.* The yield point of structural steel is a little above the proportional elastic limit. It is best to base the factor of safety on the proportional elastic limit.

Within the elastic limit these formulas are experimentally and theoretically correct. When the dimensions of the column are given and the eccentricity is known, equation (13) gives the true unit stress, and may be used to determine whether a given column will carry a given load with safety.

137. Application of Column Formulas.—When it comes to computing the total load P which a given column will carry with a certain allowable unit stress, or to designing a column for a given load, these equations are not convenient, since neither of these quantities is expressed explicitly. Such problems must be solved by the method of trial and error.

When a number of such problems are to be solved, it is a great saving of time to represent equation (15) by means of curve or table. To determine the relative values of $\frac{P}{A}$ and $\frac{l}{r}$ which makes the maximum unit stress in the concave side of the column at the point of maximum deflection equal to the ultimate strength of the material, equation (15) may be written:

$$\frac{ec}{r^2} \sec \sqrt{\frac{P}{AE}}\frac{l}{2r} = \frac{s_u}{\frac{P}{A}} - 1 \tag{1}$$

where s_u is the ultimate strength of the material. It is difficult to solve for $\frac{P}{A}$ corresponding to a given value of $\frac{l}{r}$, but it is easy to solve for $\frac{l}{r}$ corresponding to a given $\frac{P}{A}$

Table XII gives most of the calculation for structural steel for which E is 29,000,000 pounds per square inch and the ultimate strength is taken as 36,000 pounds per square inch, for the case where the eccentricity is such that $\frac{ec}{r^2} = 0.2$.

* See tests of columns for the Pennsylvania Railroad, by BUCHANAN, *Engineering News*, Dec. 26, 1907.

TABLE XII.—ULTIMATE UNIT LOAD ON A COLUMN WITH ROUND ENDS

$E = 29{,}000{,}000$ and $s_u = 36{,}000$ pounds per square inch; $\frac{ec}{r^2} = 0.2$.

$\frac{P}{A}$	$0.2 \sec \sqrt{\frac{P}{AE}}\frac{l}{2r}$	$\sec \sqrt{\frac{P}{AE}}\frac{l}{2r}$.	$\sqrt{\frac{P}{AE}}\frac{l}{2r}$ Degrees	$\sqrt{\frac{P}{AE}}\frac{l}{2r}$ Radians	$\frac{l}{r}$
1,000	35.0	175.	89° 40′	1.565	533.
3,000	11.0	55.	88° 58′	1.553	305.
6,000	5.0	25.	87° 42′	1.531	213.
10,000	2.6	13.	85° 35′	1.494	161.
15,000	1.4	7.	81° 47′	1.427	126.
18,000	1.0000	5.000	78° 28′	1.370	110.
20,000	0.8000	4.000	75° 31′	1.318	100.
22,000	0.6363	3.182	71° 41′	1.251	90.8
24,000	0.5000	2.500	66° 25′	1.159	80.6
25,000	0.4400	2.200	62° 58′	1.099	76.6
26,000	0.3846	1.923	58° 40′	1.024	68.4
27,000	0.3333	1.667	53° 08′	0.927	60.8
28,000	0.2857	1.429	45° 34′	0.795	51.2
29,000	0.2414	1.207	34° 03′	0.594	37.6
30,000	0.2000	1.000	0° 0′	0.000	0.0

To find the value of $\frac{l}{r}$ which makes the maximum unit stress 36,000 pounds per square inch when the unit load is 15,000 pounds per square inch, with $\frac{ec}{r^2} = 0.2$,

$$\frac{36{,}000}{15{,}000} - 1 = 1.4 = 0.2 \sec\left(\sqrt{\frac{15{,}000}{29{,}000{,}000}}\frac{l}{2r}\right). \tag{1}$$

$$\sec\left(\sqrt{\frac{15}{29{,}000}}\frac{l}{2r}\right) = 7. \tag{2}$$

$$\sqrt{\frac{15}{29{,}000}}\frac{l}{2r} = 81^\circ\ 47' = 1.427 \text{ radians.} \tag{3}$$

$$\frac{l}{r} = 2 \times 1.421\sqrt{\frac{29{,}000}{15}} = 126. \tag{4}$$

Curve I of Fig. 167 is drawn from the data of Table XII. Curve II of this figure is for the case when the eccentricity is zero, and is called Euler's curve.

The two curves approach each other for large values of $\frac{l}{r}$ where the amount of eccentricity makes little difference. For

small values of $\frac{l}{r}$ the eccentricity makes a great difference in maximum unit stress.

For values of $\frac{ec}{r^2}$ less than 0.2, other curves could be drawn between curves I and II, and below the horizontal line at 36,000 pounds per square inch.

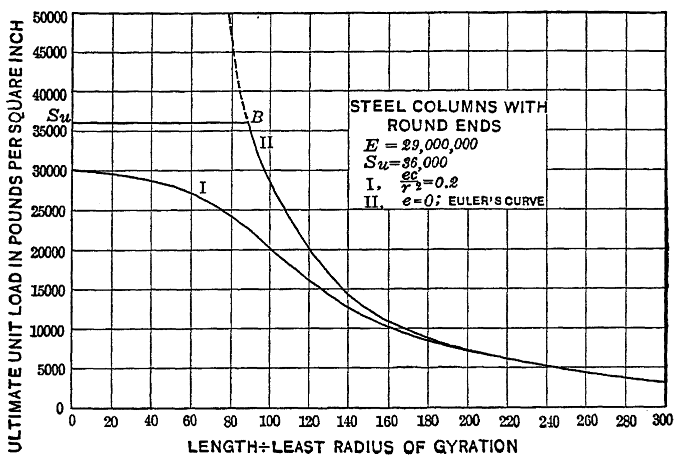


FIG. 167.—Ultimate unit load on steel column.

Example

Use curve I of Fig. 167 to find the I-beam 10 feet long as a column with round ends to carry a load of 40,000 pounds with a factor of safety of 2.5.

The ultimate load with a factor of safety of 1 is $40{,}000 \times 2.5 = 100{,}000$ pounds. As the unit load cannot be over 30,000 pounds per square inch the area must be greater than $100{,}000 \div 30{,}000$ or 3.33 square inches. But with an I-beam of this section r is small, making $\frac{l}{r}$ large and $\frac{P}{A}$ relatively small, so that it is advisable to begin with an area twice as great. A 9-inch 21-pound I-beam has an area of 6.31 square inches and a least radius of gyration of 0.90 inch, from which $\frac{l}{r} = 133$ and $\frac{P}{A}$, from the curve, is 13,750 pounds per square inch.

$$13{,}750 \times 6.31 = 86{,}700 \text{ pounds.}$$

With the 10-inch 25-pound I-beam, the area is 7.37 square inches; $r = 0.97$ inch; $\frac{l}{r} = 123$; $\frac{P}{A} = 15{,}500$ pounds per square inch.

$$15{,}500 \times 7.37 = 114{,}000 \text{ pounds.}$$

The first of these beams is too small, and the second is larger than necessary. The 9-inch 25-pound I-beam would come nearer but as it is as heavy and expensive as the 10-inch beam, the latter would be chosen.

138. Euler's Formula.—As the secant of 90 degrees is infinite any column will fail if

$$\sqrt{\frac{Pl^2}{4\,EI}} = \frac{\pi}{2},$$

from which

$$P = \frac{\pi^2 EI}{l^2},$$

$$\frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}. \qquad \text{Formula XXVI.}$$

Formula XXVI and the equation above it are two ways of writing Euler's Formula. It may be derived directly from equation (1) of Article 136 for the case where the eccentricity is zero.

$$EI\frac{d^2y}{dx^2} = -Py. \tag{1}$$

Multiply by dy,

$$EI\frac{d^2y}{dx^2}\,dy = -Py\,dy. \tag{2}$$

$$\frac{d^2y}{dx^2}\,dy = \frac{dy}{dx}\frac{d^2y}{dx} = z\,dz \text{ where } z = \frac{dy}{dx}.$$

$$\frac{EI}{P}z\,dz = -y\,dy. \tag{3}$$

$$\frac{EIz^2}{P} = -y^2 + C_1^2. \tag{4}$$

$$\sqrt{\frac{EI}{P}}\frac{dy}{dx} = \sqrt{C_1^2 - y^2}. \tag{5}$$

$$\sqrt{\frac{P}{EI}}\,dx = \frac{dy}{\sqrt{C_1^2 - y^2}}. \tag{6}$$

$$\sqrt{\frac{P}{EI}}\,x = \sin^{-1}\frac{y}{C_1} + C_2. \tag{7}$$

$y = 0$ when $x = 0$, hence $C_2 = 0$ (or $n\pi$).
Using $C_2 = 0$,

$$\sin\sqrt{\frac{P}{EI}}\,x = \frac{y}{C_1}. \tag{8}$$

When $x = l$, $y = 0$,

$$\sin\sqrt{\frac{P}{EI}}\,l = 0;$$

$$\sqrt{\frac{P}{EI}}\,l = \pi \text{ (or } n\pi);$$

$$P = \frac{\pi^2 EI}{l^2}. \qquad \text{Euler's Formula.}$$

P is the load requd. to bend the column
or breaking load
i.e. the safe load divide this by factor of safety.

Equation (8) shows that the column bends as a sine curve. Formula XXVI contain I but does not include the distance to the outer fiber. It follows, that when the *eccentricity is negligible and the ratio of the length to the least radius of gyration* is large, the value of the ultimate load does not depend upon the form of the column, except in so far as the form changes the moment of inertia.

Curve II of Fig. 167 is Euler's curve for a modulus of elasticity of 29,000,000 pounds per square inch. As a mathematical curve it is of infinite length. As an engineering curve it must not be used above the point B where the unit load is the elastic limit of the material.

It will be shown later that it is best to use Euler's curve to only one-third the elastic limit.

Example

Find the total load with a factor of safety of 2 on a round steel rod 2 inches in diameter, if the elastic limit is 30,000 pounds per square inch and E is 29,000,000 pounds per square inch, for lengths of 40, 60, 80, and 100 inches.

As these lengths are all multiples of 20 inches, begin with this length and find the others by dividing by the square of the ratio. The radius of gyration of a solid circular area being one-half the radius, $r = \frac{1}{2}$ inch. For $l = 20$ inches,

$$\frac{P}{A} = \frac{9.87 \times 29{,}000{,}000}{1{,}600} = 178{,}894 \text{ pounds per square inch}$$

Length, inches	Unit load, lb./in.^{2}	Total safe load, pounds
40	44,723	......
60	19,877	31,223
80	11,181	17,563
100	7,156	11,240

For the 40-inch length $\frac{P}{A}$, as calculated by Euler's Formula, is 44,700 pounds per square inch, which is above the elastic limit, and therefore cannot be used. For the 60-inch length the unit load is less than 20,000 pounds per square inch. This is below the elastic limit, and may be divided by the factor of safety and multiplied by the area of the section to get the total safe load. Euler's formula should not be used for this load unless it is certain that the eccentricity is negligible. For the greater lengths the eccentricity makes little difference, as may be seen from the curves of Fig. 167.

Problems

1. A yard stick, with the ends slightly rounded, was placed vertical with the lower end on a platform scale and a load was applied to the upper end (Fig. 165). The load and deflection were measured.

Load in pounds	Deflection at the middle, in inches	
5.00	0.03	
6.00	0.20	
6.40	0.25	
6.48	1.00	(Load dropped to 6.28)
6.28	2.50	

Calculate EI from the last two readings by Euler's formula. *Ans.* 851, 825.

2. The yard stick of Problem 1, supported as a beam at points 34 inches apart, was deflected $^{31}\!/_{32}$ inch at the middle by a load of 1 pound at the middle. Find EI and compare the result with Problem 1.

3. The yard stick above mentioned was 1.06 inches wide and 0.18 inch thick. Find E and $\frac{l}{r}$.

4. Find the total safe load, with a factor of safety of 4, on a 6-inch by 6-inch oak post, 15 feet long, if E is 1,000,000 pounds per square inch and the elastic limit is 3,000 pounds per square inch. *Ans.* 8,225 pounds.

5. Could Euler's Formula be used in Problem 4 for a length of 5 feet?

139. Classification of Columns.—Columns may be divided, according to the nature of the ends, into the following classes:

I. Both ends free to turn about horizontal axes but not free to move laterally, Figs. 166 and 168, I.

II. One end fixed and the other end free to turn and free to move laterally, Fig. 168, II.

III. Both ends fixed so that the tangents at the ends do not change, Fig. 168, III.

IV. One end fixed and the other end free to turn about one or more horizontal axes, but not free to move laterally, Fig. 168, IV.

Case I is the only one so far considered. If L is the total length of the column, and l is the length of the sine curve ABC as used in the theory of Articles 136 and 138, $l = L$ for case I.

In case II, the entire column of length L corresponds to the upper half AB of the sine curve. Hence for case II we use $2L$ for l in Formulas XXV and XXVI.

In case III, $\frac{dy}{dx}$ is zero at each end and at the middle. The middle half ABC corresponds to the sine curve of case I. This portion of the sine curve is represented by l in the formulas. If L is the entire length DF, then $l = \frac{L}{2}$. A column with both

ends rigidly fixed will carry as great a load as a column of half its length with ends free to turn.

The points A and C, at one-fourth the length from the ends, are points of counterflexure. The portion AD is one-half of a sine curve. If revolved 180 degrees in the plane of the paper about the point A, the curve AD will coincide with AB. The moment is zero at A and C. Case IV is fixed at one end and free to turn at the other, but not free to move laterally. The point

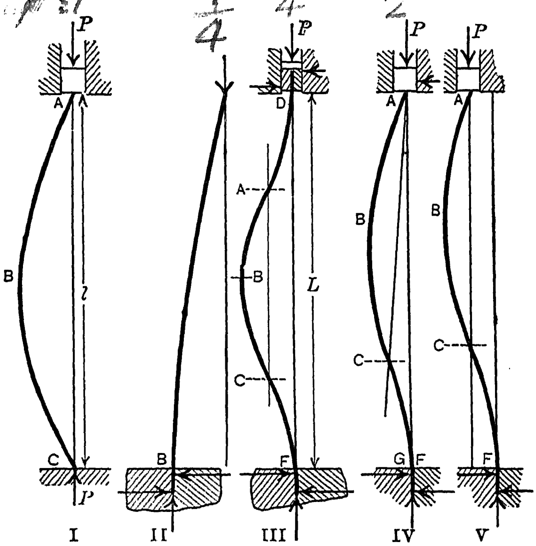


FIG. 168.—Types of ideal columns.

of counterflexure is at C. As there is no moment at C, the resultant force at A must be in the direction AC. The portion ABC forms a sine curve similar to the preceding cases with the line AC corresponding to the X axis in Fig. 166. The lower portion CF forms a part of a sine curve as far as the plane of the body which holds it. Below that plane it is straight. The portion CG is less than one-half of AC. It is evident, therefore, that AC is more than two-thirds of L. The solution of the differential equation shows that AC is nearly 0.7 L.

$$l = 0.7\,L, \qquad l^2 = 0.5\,L^2 \text{ nearly.}$$

In Fig. 168, V, the top of the column has been displaced laterally. If this displacement is such that the point B is as far from the line AC as the top A is displaced from the vertical line

through F, then the line AC from the end to the point of counterflexure becomes vertical. In this position AC is two-thirds of L, there is no horizontal force at the top, and the vertical force P is greater than in Fig. 168, IV. The position is unstable and easily changes to the one in which the curves are reversed, with C and B deflected to the right of the vertical line through F, in which position the load P, which produces a large deflection, is less than in case IV.

Problems

1. A yard stick, with ends rounded, was deflected a large amount by a load of 6.1 pounds at the end, as in case I. Find EI by Euler's formula.

2. The same yard stick was clamped 4 inches from one end and the load was applied as in Fig. 168, IV. It was found that a load of 15.42 pounds produced a deflection of over 1.5 inches. Find EI by Euler's formula.

3. The load in Problem 2 was displaced 1 inch south of the vertical line through the bottom. The vertical component of this load was 17.12 pounds with a deflection of 2 inches south. The horizontal component was found to be zero. Find EI, using two-thirds of 32 inches as l.

4. A solid circular steel rod stands in a vertical position with the lower end fixed. A load of 100,000 pounds is applied at the free upper end at a distance of 1 inch from the center. The diameter of the rod is 6 inches, and its length from the fixed point is 15 feet. If E is 30,000,000, find the deflection at the end and the maximum fiber stress by the formulas of Article 136. *Ans.* Maximum stress, 21,320 pounds per square inch.

140. End Conditions in Actual Columns.—The classification of Article 139 represents *ideal* conditions, which are only approximated in practice. The columns in actual use are:

Round-end columns, which end with spherical or cylindrical surfaces. They sometimes end with knife-edges, which may be regarded as cylinders of small radius. The *round* surfaces roll on *plane* surfaces with practically no friction. Round-end columns are not used in structures and are rarely used in machines. They are used in tests to check the accuracy of theory, as they fulfill very closely the conditions of case I of ends free to turn.

A *pin-end* or *hinged-end column* ends with cylindrical surfaces which turn in *cylindrical bearings* (Fig. 169, I). Fig. 169, II, shows one end of a pin-connected column made of two channels latticed together. This form of connection is commonly used in bridges. A column which ends with a ball and socket is practically the same as a hinged-end column, except that it is free to turn in any plane instead of in the single plane normal to the axis of the hinge.

If the pin of a hinged-end column rolled on a plane surface, there would be little friction, and the case would be the same as that of the round-end columns. Usually the pin turns in a close-fitting bearing, so that the friction is considerable. A hinged-end column may be anywhere between case I, with the ends free to turn, and case III, with the ends fixed. If the pin is small, the moment arm of the friction is small, and a slight eccentricity will cause it to turn. If the pin is large, the opposite is true. In the moving parts of machinery, the pin connections are lubricated so that they turn easily. The connecting rod of an engine is an example.

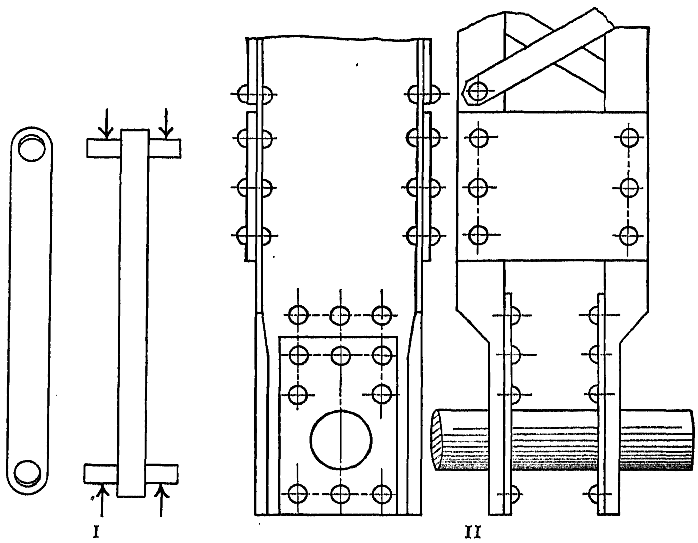


FIG. 169.—Pin-end columns.

Fig. 170 shows, diagrammatically, the behavior of a pin-end column. At first it acts as a column with fixed ends (Fig. 170, I). When the moment at the end becomes greater than the product of the starting friction at the surface of the pin multiplied by its radius, the column turns at the end to some position similar to Fig. 170, II. In this position, the points of counterflexure A and C are nearer the ends, and the moment on the pins is less. The column may finally change to the position of Fig. 170, III, which is that of case I. In this last position, it will support a load much smaller than in the first position. If the ratio of the length to least radius of gyration is 200 or more, so that Euler's

formula applies to both the whole and the half length, the column in the last position will carry a load only one-fourth as great as in the first one.

Some interesting tests of columns were made at the Pencoyd Iron Works in 1883 by James Christie.* In these tests, some of the so-called *hinged*-end columns were fitted with hemispherical balls turning in sockets. The balls were located as nearly as possible in the line of the axis of the column by careful measurements. Owing to the fact that no column is absolutely straight and perfectly uniform in section and homogeneous in structure throughout its entire length, this method did not always put the centers of the hemispheres exactly on the axis of the column.

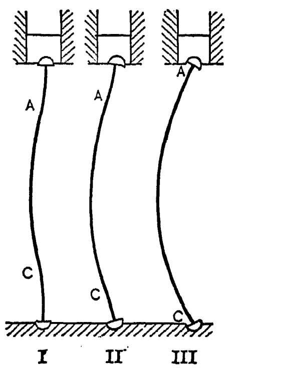


FIG. 170.—Deflection of hinged-end column.

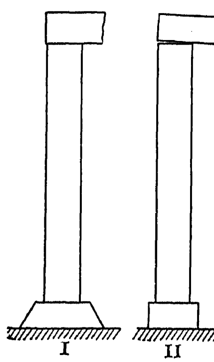


FIG. 171.—Square-end columns.

The final adjustment was determined by trial in the testing machine; a small load was applied and the deflection measured. The hemispheres were then moved a little and the test repeated, until a position was found where a considerable load caused no appreciable deflection. The column was then loaded to failure.

†"When the point of greatest strength was reached, the behavior of the specimen was peculiar. Under ordinary circumstances the bar, while bending under strain, rotated from the start on its hinged ends. When correctly centered, no such rotation occurred at the beginning of the deflection, but the bar bent like a flat-ended strut, till the point of failure was reached,

* *Transactions of the American Society of Civil Engineers,* 1883, pages 85–122.

† *Ibid.,* page 87.

when it rotated on its ends suddenly, as sometimes to spring from the machine." These results could not be secured when the balls or pins rolled on plane surfaces, and were difficult to get when the pins were small.

The effect of the size of the pin was shown in these experiments. Two angles of the same length were cut from the same bar. One of these tested with a 2-inch ball and socket failed at 36,500 pounds per square inch; the other tested with a 1-inch ball and socket failed at 24,010 pounds per square inch. Similar results were obtained in other experiments.

These tests and many others show that the friction at the ends of a hinged-end column partially fixes the ends and increases the ultimate strength. It must be remembered, however, that in the testing machine the loads are applied with little vibration. In structures such as railway bridges, where there is large vibration, it is probable that the friction of the pins gives little help, and it is safest to regard hinged-end columns as equivalent to round-end columns.

Square-end or *flat-end* columns end with plane surfaces in contact with plane surfaces. They must be accurately fitted if eccentric loading is to be avoided. If a beam resting on a square-end column bends under its load (Fig. 171, II), the load on the column becomes eccentric. Footings which support columns often settle unevenly and cause large eccentricity.

Pin-end columns are practically square-end with respect to the axis of the pin.

A column with a pin connection at one end and a square connection at the other is called a *pin*-and-*square* column. This term includes columns with one end *fixed* and the other hinged. This column approximates the conditions of case IV.

Fixed-end columns are riveted to the remainder of the structure in buildings and bridges. In machines they are fastened in various ways. The connection can never be absolutely rigid, and the member to which the column is fixed must suffer some deflection, so that there is always some change in the slope of the tangent at the "fixed" points. When the column is very flexible compared with the body to which it is fixed, it may then be regarded as an example of the ideal case and l may be taken as equal to half the length L. In the case of the yard stick described in Problem 2 of Article 139, the column was firmly clamped to

the 2-inch by 4-inch post and the value of $\frac{L}{r}$ was over 800, making it relatively very flexible, so that this gave consistent results when treated as an example of a column fixed at one end. In machines this condition is sometimes met, but it never occurs in structures.

141. Some Experiments Showing Effect of End Conditions.—It is evident that the value of l to be used with "square-" and "fixed-" end columns in calculating the unit load is greater than $\frac{L}{2}$ and less than L. In the case of the "pin-and-square" column,

TABLE XIII.—PENCOYD TESTS OF WROUGHT-IRON STRUTS
Average Results for Angles and Tees

$\frac{L}{r}$	$\frac{P}{A}$, ultimate unit load in pounds per square inch			
	Round ends	Hinged ends	Flat ends	Fixed ends
20	44,000	46,000	49,000	45,000
40	36,500	40,500	41,000	38,000
60	30,500	36,000	36,500	34,000
80	25,000	31,500	33,500	32,000
100	20,500	28,000	30,250	30,000
120	16,500	24,250	26,500	28,000
140	12,800	20,250	23,250	25,500
160	9,500	16,350	20,500	23,000
180	7,500	12,750	18,000	20,000
200	6,000	10,750	15,250	17,500
220	5,000	8,750	13,000	15,000
240	4,300	7,500	11,500	13,000
260	3,800	6,500	10,250	11,000
280	3,200	5,750	8,750	10,000
300	2,800	5,000	7,350	9,000
320	2,500	4,500	5,750	8,000
340	2,100	4,000	4,650	7,000
360	1,900	3,500	3,900	6,500
380	1,700	3,000	3,350	5,800
400	1,500	2,500	2,950	5,200
420	1,300	2,250	2,500	4,800
440	......	2,100	2,200	4,300
460	......	1,900	2,000	3,800
480	......	1,700	1,900	......

it is less than L and (if the friction of the pin is small) greater than $0.7\,L$. The best values to be used should be determined from tests of full-size columns under a wide range of conditions.

The experiments of Christie, previously mentioned, are instructive in this regard. In Table XIII are the results of these experiments for angle and tee sections.

In this table L is the total length of the column as in Fig. 146, and r is the least radius of gyration. It was found that the columns failed in the direction for which radius of gyration was the minimum.

The figures of this table give some idea of the relative value of hinged, round, fixed, and flat ends. For instance with the unit load of 25,000 pounds per square inch, $\frac{L}{r}$ is 80 for round ends. For flat ends, this value of the unit load lies between 26,500 for which $\frac{L}{r}$ is 120 and 23,250 for which $\frac{L}{r}$ is 140. Interpolation gives 129 as the value of $\frac{L}{r}$ for flat ends corresponding with the ultimate unit load of 25,000 pounds per square inch. As far as this experiment goes, it indicates that in calculating a flat-end column the value of l in the formulas should be taken as $\frac{80\,L}{129} = 0.62\,L$. In a similar way for fixed ends the unit load of 25,000 pounds per square inch corresponds with an $\frac{L}{r}$ of 144, which makes $l = 0.56L$ for this particular case. For hinged ends the corresponding *slenderness ratio* is 119 and $l = 0.67L$.

In the case of the hinged ends, owing to the lack of vibration, the load was probably greater than would be found in railway bridges subjected to the jar of fast trains. A lubricated hinged-end column, such as the connecting rod of an engine, would probably approximate closely to an ideal round-end column, and l would be nearly equal to L in the formulas.

Problems

1. Using $\frac{L}{r}$ equals 60 for round ends, find the equivalent lengths of hinged, flat, and fixed ends, and the corresponding values of l in terms of the entire length L. *Ans.* $l = 0.70\,L$, $l = 0.61\,L$, $l = 0.63\,L$.

2. Using $\frac{L}{r}$ equals 100 for round-end columns, find the corresponding values for hinged, flat, and fixed ends, and values of l in terms of L.

Ans. $\frac{L}{r} = $ **138, 160, 177.**

If we take all the values for round ends from 40 to 200 inclusive and determine the values of $\frac{L}{r}$ which give the same unit load for the other end conditions, we get the following ratios:

	Hinged	Flat	Fixed
Minimum	1.29	1.50	1.25
Maximum	1.45	1.69	1.87
Mean of all	1.37	1.60	1.72

In the case of the fixed ends, only one value was below 1.50.

As far as these figures go, they indicate that a flat-end column 16 feet long, a fixed-end column 17.2 feet long, or a hinged-end column 13.7 feet long, will carry the same total load as a round-end column 10 feet long of the same cross-section.

142. The Ultimate Strength.—The formulas of the preceding articles are derived under the assumption that E is constant. They are valid, therefore, to the proportional elastic limit only. The actual ultimate strength, for columns of one piece, is considerably above this limit. Table XIII for wrought-iron columns made of single shapes show an ultimate strength of 49,000 pounds per square inch. These tests were made slowly so there was ample time for the raising of the yield point which occurs in wrought iron and steel.

For built-up columns of wrought iron or steel the ultimate strength is the yield point of the material.

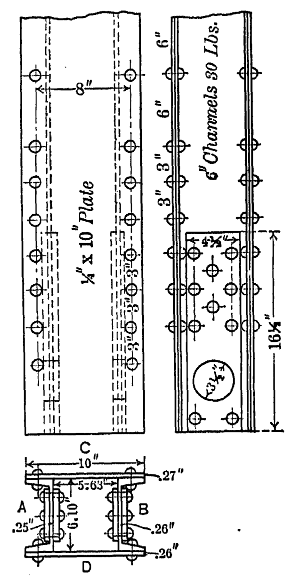


FIG. 172.—Column tested at watertown arsenal.

Fig. 172 shows one of a series of wrought-iron columns tested at Watertown Arsenal in 1884 ("Tests of Metals," 1884, page 17). The column was tested with 3.5-inch pins. The length center to center of pins was 20 feet, and the deformation was measured in a gage length of 200 inches. The average cross-section of channels and plates was determined from the weight and specific gravity. The cross-sections were:

	Square inches
Channel A	3.00
Channel B	3.05
Plate C	2.66
Plate D	2.60
	11.31

TABLE XIV.—TEST OF WROUGHT-IRON PLATE AND CHANNEL COLUMN AT WATERTOWN ARSENAL

Total load	Compression in gage length of 200 inches	Deflection at the middle	
		Perpendicular to pins	Parallel to pins
Pounds	Inch	Inch	Inch
5,000	0.0	0.0	0.0
30,000	0.0169	0.0	0.0
50,000	0.0293	0.01	0.01
5,000	0.0	0.0	0.0
80,000	0.0482	0.01	0.01
100,000	0.0610	0.02	0.01
5,000	0.0	0.0	0.0
130,000	0.0804	0.03	0.02
150,000	0.0931	0.03	0.02
5,000	0.0010	0.0	0.01
180,000	0.1118	0.04	0.03
200,000	0.1247	0.04	0.03
5,000	0.0016	0.0	0.02
230,000	0.1444	0.06	0.03
250,000	0.1580	0.07	0.03
5,000	0.0041	0.0	0.03
260,000	0.1651	0.09	0.03
270,000	0.1725	0.10	0.03
280,000	0.1797	0.12	0.03
290,000	0.1870	0.13	0.03
300,000	0.1954	0.17	0.03
5,000	0.0110	0.03	0.03
310,000	(*Micrometer*	0.20	0.03
320,000	*removed*)	0.27	0.03
325,000	....	0.32	0.03
330,000	....	0.45	0.03
330,100	....	0.48	0.03

Failed by deflection perpendicular to the plane of the pins; with plate C on the convex side.

"After reaching the maximum load, the deflection increased slowly till it reached 0.75 inch, the load at the time being 320,000 pounds. From this point the rate of deflection accelerated till it reached 1.80 inches under 310,000 pounds load, when sudden springing occurred, increasing the deflection to 3.35 inches, while the pressure fell to 155,000 pounds.

"Released to the initial load, the deflection was 2.08 inches.

"A sharp bend was found 20 inches from the middle of the post; the plate on the concave side buckled between the riveting. Pinholes elongated 0.01 inch."

The initial load was 5,000 pounds. The set was determined by returning to the initial load after each 50,000 pounds increment. The deflection at the middle was measured perpendicular and parallel to the pins. Some of the readings are given in Table XIV.

In this test the maximum load was nearly 29,200 pounds per square inch. The unit stress on the concave side was this figure plus the stress due to bending. We get the maximum bending moment by multiplying the load by the deflection 0.48 inch. The moment of inertia of the section with respect to an axis parallel to the pins is about 86. This gives a bending stress of 6,100 pounds, making the total compressive stress at the beginning of failure 35,300 pounds per square inch.

It is probable that at the beginning of failure the column was in condition I of Fig. 170, the friction causing it to act as if the ends were fixed. In this case the moment arm is only half the deflection at the middle and the actual maximum stress is only 32,000 pounds per square inch.

Other columns of the same set gave similar results. Other sets of tests, notably those of Buchanan for the Pennsylvania Railroad, agree in indicating that the value of s_u should not exceed 35,000 for wrought iron and 40,000 for structural steel.

143. Column Formulas Based on Allowable Stress.—Instead of deriving the curve of unit load and slenderness ratio from the ultimate strength and modulus of elasticity, it may be based upon the allowable unit stress and on a *working modulus* of elasticity, obtained by dividing E by the factor of safety. In the case of structural steel, when 16,000 pounds per square inch is used as the allowable unit stress in compression the factor of safety is about 2.25 based on a yield point of about 36,000 pounds per square inch. Dividing 29,000,000 by 2.25 gives practically 13,000,000 pounds per square inch as the working modulus of elasticity. Representing the working modulus of elasticity by E_w and the allowable unit stress by s_w, equation (1) of Article 137 becomes

$$\frac{ec}{r^2} \sec \sqrt{\frac{P}{AE_w}}\frac{l}{2r} = \frac{s_w}{\frac{P}{A}} - 1.$$

Fig. 173 is the curve giving the allowable values of the unit load for soft steel.

It is not necessary that the same factor of safety be used in

calculating s_w and E_w. This depends largely upon the variation of the quantities upon which the factor of safety is based, and upon whether the factor of safety is employed chiefly to provide for variation of the resisting material or for variation of loading. In structural steel there is little variation in the modulus of elasticity. The yield point varies somewhat more relatively but the difference is not large in steel of a given quality.

In cast iron there is no yield point and the proportional elastic limit is low. Fig. 96 shows it to be about 13,000 pounds per square inch in compression and 3,000 pounds per square inch, or

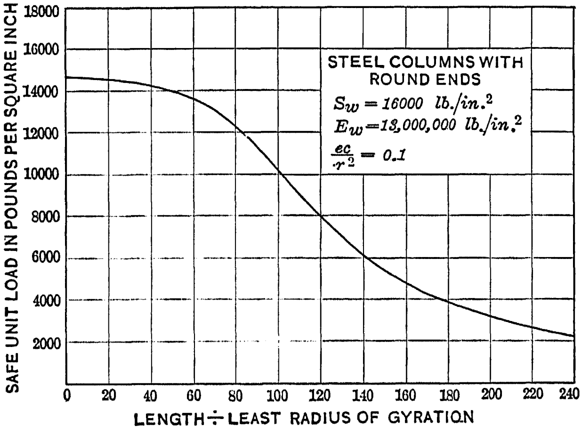


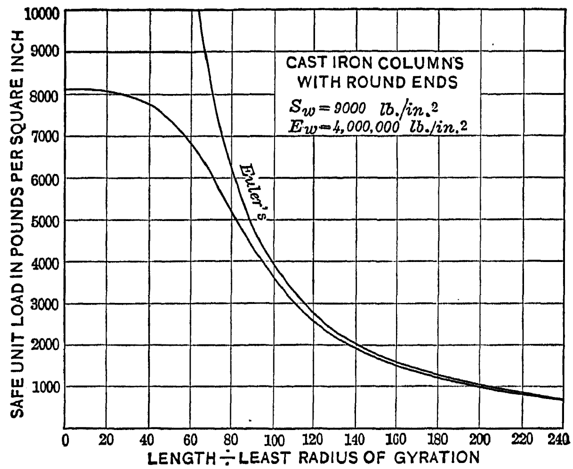
FIG. 173.—Allowable unit load on steel column.

less, in tension. The ultimate strength in tension is about 25,000 pounds per square inch, so that if there is much eccentricity the unit load on a cast-iron column should be kept low.

The ultimate compressive strength of cast iron is about 80,000 pounds per square inch. The value of s_w which is recommended by some standard specifications is 10,000 or 9,000 pounds per square inch, corresponding to a factor of safety (based on the ultimate strength) of 8 or 9. This is unnecessarily high for the modulus of elasticity. A factor of safety of 4 makes the working modulus of elasticity 4,000,000 pounds per square inch. The curve of Fig. 174, I, is based on this modulus and on an allowable compressive stress of 9,000 pounds per square inch.

Fig. 174, II, is the curve for long-leaf yellow pine. The working values of the unit compressive stress and the modulus of elasticity are taken as 1,000, and 300,000 pounds per square inch

corresponding to a factor of safety of 5 in each case. The great variation in the modulus of elasticity of timber makes it neces-



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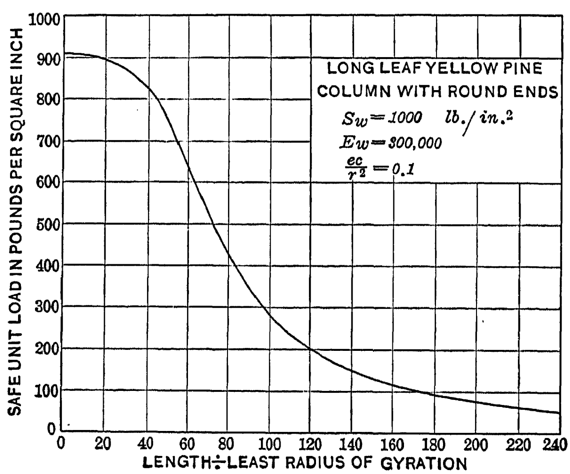

II

FIG. 174.—Allowable unit loads on columns.

sary to use a greater factor of safety, than would be required in material having a constant modulus.

CHAPTER XIV

COLUMN FORMULAS USED BY ENGINEERS

144. Straight-line Formulas.—The curves of Fig. 167 show that Euler's formula may be used with little error when $\frac{l}{r}$ is large and that a considerable eccentricity makes little difference. For smaller values of the slenderness ratio, Euler's formula *must not be used,* and a slight difference in the eccentricity makes a relatively large difference in the results of the secant formula. In structures, especially where flat-end or fixed-end columns are used, there is usually considerable uncertainty in regard to the amount of eccentricity. It is, therefore, not worth while to go through the labor of calculating with the secant formulas, except in the cases of relatively large known eccentricity. Engineers make use of simpler approximate formulas. A few years ago Rankine's formula was most used. At present, the *straight-line formulas* have the preference in American practice.

A straight-line column formula for the ultimate unit load has the form:

$$\frac{P}{A} = s_u - k\frac{l}{r}, \qquad \text{Formula XXVII.}$$

where k is a constant depending upon the properties of the material. If $\frac{P}{A} = y$ and $\frac{l}{r} = x$, this is recognized as the equation of a straight line with the Y intercept equal to s_u and with a negative slope equal to k.

If in Fig. 167 a straight line be drawn through the point $(0, s_u)$ and tangent to Euler's curve, this straight line is found to deviate little from the secant curve. Except for small values of $\frac{l}{r}$, slight changes in the eccentricity will cause the secant curve to pass from one side of this straight line to the other. Such a straight line, then, will give fairly approximate values for the unit loads for the uncertain eccentricities which occur in practice, for all values of $\frac{l}{r}$ to the left of the point of tangency except very small ones.

Fig. 175 shows the method of finding the constant of the straight-line formula graphically. Curve I is Euler's curve for steel for which E is 29,000,000 pounds per square inch. The straight line II is drawn tangent to Euler's curve and passes through point s_u on the Y axis. With s_u equal to 36,000 pounds per square inch the straight line intercepts the X axis at about 233. The slope is 36,000 ÷ 233 = 154, so that the straight-line formula for this steel is $\frac{P}{A} = 36{,}000 - 154\frac{l}{r}$.

This straight-line formula is valid to the point of tangency, or a little beyond that point where it does not differ appreciably from

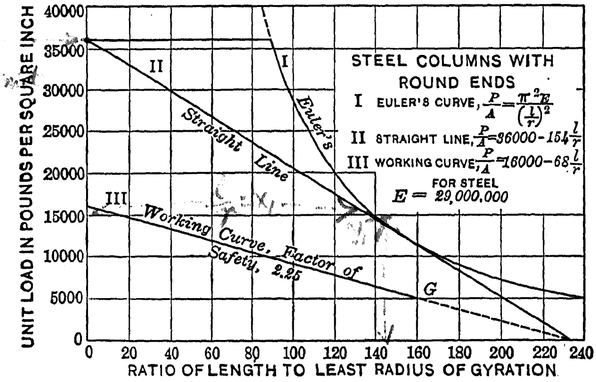

FIG. 175.—Straight line for steel column.

Euler's curve. It may be used for values of $\frac{l}{r}$ up to about 160; and Euler's equation is used beyond that point.

Curve III is the straight line for allowable values of $\frac{P}{A}$ with a factor of safety of 2.25. It ends at the point G. Beyond that point use Euler's and divide the result by 2.25.

Problem

Plot Euler's curve for timber for which E = 1,200,000 pounds per square inch. Draw a straight-line if ultimate strength is 4,000 pounds per square inch, and derive a working formula with a factor of safety of 4.

145. Algebraic Derivation of the Straight-line Formulas.—While a straight-line formula may always be derived graphically by plotting Euler's curve and drawing the tangent, the methods of Calculus are convenient and lead to a simple algebraic result.

The problem is that of drawing a straight line through a given point tangent to a given curve. Euler's formula may be written:

$$y = \frac{a}{x^2}, \tag{1}$$

where

$$y = \frac{P}{A}, x = \frac{l}{r}, \text{ and } a = \pi^2 E.$$

The problem is to draw a tangent to curve (1) which shall pass through the point $(0, s_u)$. The equation of this tangent line is:

$$y = -\frac{2a}{x_1^3}x + s_u, \tag{2}$$

where x_1 is the abscissa of the point of tangency.

Since the straight line (2) passes through the point of tangency whose coördinates are (x_1, y_1), these coördinates satisfy the equation of the line; hence

$$y_1 = -\frac{2a}{x_1^2} + s_u. \tag{3}$$

Also, since the point of tangency is on the curve, these coördinates satisfy equation (1); and

$$y_1 = \frac{a}{x_1^2}, \tag{4}$$

Combining (3) and (4) for the coördinates of this point of contact:

$$y_1 = \frac{s_u}{3}, \qquad \text{Formula XXVIII} \tag{5}$$

$$x_1^2 = \frac{3a}{s_u}; \tag{6}$$

The value of x_1 from (6) may be substituted in (2) to get the desired straight-line equation. It is better to use the easily remembered fact of Formula XXVIII that the ordinate of the point of tangency is one-third of the Y intercept of the straight line. This value substituted in Euler's formula gives the abscissa of the point of contact. This gives the coördinates of two points on the straight line from which to derive its equation.

Example

Derive a straight-line formula for steel for which the ultimate strength is 36,000 pounds per square inch and E is 29,000,000 pounds per square inch.

One-third of the ultimate strength is 12,000 pounds per square inch, which substituted in Euler's formula gives

$$12{,}000 = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2};$$

$$\left(\frac{l}{r}\right)^2 = \frac{9.87 \times 29{,}000{,}000}{12{,}000};$$

$$\frac{l}{r} = 154.$$

$$k = \frac{\frac{2}{3} \times 36{,}000}{154} = 156;$$

$$\frac{P}{A} = 36{,}000 - 156\frac{l}{r}.$$

A working formula with a factor of safety of 2.25 is:

$$\frac{P}{A} = 16{,}000 - 69\frac{l}{r}.$$

Problems

1. Derive a straight-line formula for timber for which the ultimate strength is 3,000 pounds per square inch and the modulus of elasticity is 1,200,000 pounds per square inch. *Ans.* slope $= 2{,}000 \div 109 = 18.$

$$\frac{P}{A} = 3{,}000 - 18\frac{l}{r}.$$

2. Derive a working formula with a factor of safety of 3 for the material of Problem 1.

Ans. $\frac{P}{A} = 1{,}000 - 6\frac{l}{r}.$

3. Derive a straight-line formula with a factor of safety of 5 for cast iron for which s_u is 50,000 pounds per square inch and E is 15,000,000 pounds per square inch.

Ans. $\frac{P}{A} = 10{,}000 - 70\frac{l}{r}.$

Instead of deriving a straight-line formula from the ultimate strength and modulus of elasticity, it may be based on the allowable compressive stress and a working modulus of elasticity. If the same factor of safety is used in deriving the allowable unit stress and the working modulus of elasticity from the ultimate strength and E, the formula will be the same by both methods. But if the modulus of elasticity is more uniform than the ultimate strength, a lower factor of safety may be used in deriving the working modulus of elasticity than in deriving the working unit stress.

Problem

4. A given timber has an average ultimate compressive strength of 5,000 pounds per square inch but large variation in the different samples. The modulus of elasticity is 1,200,000 pounds per square inch on the average with little variation. It is believed to be equally safe if the factor of safety for the working unit stress is 5 and that for the modulus of elasticity is taken as 3. Derive a straight-line formula.

Ans. $\frac{P}{A} = 1{,}000 - 6\frac{l}{r}.$

146. Connection of Straight-line and Euler's Formulas.—A straight-line formula is valid for values of $\frac{l}{r}$ which make $\frac{P}{A}$ greater than one-third of s_u (where s_u is the first term of the second member of the equation), or which make $k\frac{l}{r}$ less than two-thirds of s_u. Since the tangent leaves Euler's curve gradually at first, the straight-line equation may be used for some distance beyond the point of tangency with little error, and this error is on the safe side.

Example

A working formula for a given steel is

$$\frac{P}{A} = 15{,}000 - 60\frac{l}{r}.$$

Find the total load on a 2-inch round rod as a column with ends free to turn for lengths of 5 feet and 8 feet. The radius of gyration is 0.5 inch and $\frac{l}{r}$ is 120 for the 5-foot column and 192 for the 8-foot column. For the 5-foot column,

$$\frac{P}{A} = 15{,}000 - 7{,}200 = 7{,}800;$$

$$P = 7{,}800 \times \pi = 24{,}500 \text{ pounds}.$$

For the 8-foot column,

$$\frac{P}{A} = 15{,}000 - 11{,}520 = 3{,}480 \text{ pounds per square inch},$$

which is less than one-third of 15,000, showing that Euler's formula should be applied to this case.

To find the limiting value of $\frac{l}{r}$,

$$10{,}000 = 60\frac{l}{r}, \quad \frac{l}{r} = 167.$$

This formula applies for values of $\frac{l}{r}$ not greater than 167.

It seldom happens that a column is made with a slenderness ratio so large that the straight-line formula cannot be used,

but when this does happen it is necessary to connect with Euler's formula. Straight-line formulas are frequently given with no hint as to the factor of safety or the modulus of elasticity upon which they should be based. The formula is frequently based upon a few experiments with relatively short columns and is likely to have too little slope, as the curve based on the secant formula is nearly horizontal at first. To extend such a formula for use with long columns, and to determine whether it is safe to so use it,

$$\frac{s_w}{3} = \frac{\pi^2 E_w}{\left(\frac{l}{r}\right)^2}, \qquad (1)$$

where s_w is the first term of the second member of the straight-line formula, E_w is a working modulus of elasticity, and $\frac{l}{r}$ is obtained from the equation of the straight line when $\frac{P}{A} = \frac{s_w}{3}$.

$$\frac{P}{A} = s_w - k\frac{l}{r},$$

$$\frac{l}{r} = \frac{2\,s_w}{3\,k};$$

which substituted in (1) gives

$$\pi^2 E_w = \frac{4\,s_w^3}{27\,k^2}. \qquad (2)$$

In the example above,

$$\pi^2 E_w = \frac{4 \times 15{,}000^3}{27 \times 60 \times 60} = \frac{25 \times 10^8}{18} \qquad (3)$$

For a length of 8 feet,

$$\frac{P}{A} = \frac{25 \times 10^8}{18 \times 192^2} = 3{,}768 \text{ pounds per square inch,}$$

which is a little greater than the results obtained from the straight-line formula.

From (3) E_w is found to be about 14,000,000 pounds per square inch. If E for this steel is known to be 29,000,000 pounds per square inch, the factor of safety is a little over 2.

Problems

1. The American Railway Engineering and Maintenance of Way Association recommends for structural steel the column formula

$$\frac{P}{A} = 16{,}000 - 70\frac{l}{r},$$

with a maximum of 14,000 pounds per square inch. What value of $\pi^2 E$ will make Euler's formula connect with this line as a tangent?

Ans. 123,800,000′

2. What is the factor of safety in Problem 1?

147. The American Railway Formula.—The Specifications of the American Railway Engineering and Maintenance of Way Association recommend for structural steel,

$$\frac{P}{A} = 16{,}000 - 70\frac{l}{r}, \qquad \text{Formula XXIX.}$$

with a maximum of 14,000 pounds per square inch. The reason for a maximum value less than 16,000 pounds per square inch is evident from Fig. 176. Fig. 176, II, shows the secant column curve

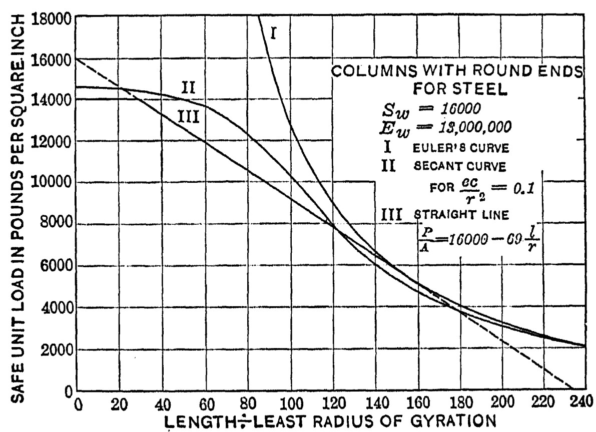

FIG. 176.—Straight line and secant curve for steel.

for material for which the allowable compressive stress is 16,000 pounds per square inch and the working modulus of elasticity is 13,000,000 pounds per square inch, corresponding to a factor of safety of about 2.25. The straight-line equation from these constants is $\frac{P}{A} = 16{,}000 - 69\frac{l}{r}$, which is practically the same as Formula XXIX. This straight line is drawn as Fig. 176, III, up to 14,000 pounds per square inch where it connects with the horizontal line $\frac{P}{A} = 14{,}000$. It will be noticed that the secant curve is nearly horizontal at first and is only little above the horizontal part of the straight line. A slight increase in the eccentricity would throw the secant curve below the straight line.

These specifications further state that, "the lengths of main compression members shall not exceed 100 times their least radius of gyration, and those for wind and sway bracing 120 times their least radius of gyration." Formula XXIX is valid for values of the slenderness ratio up to 152 so that it applies to all columns permitted in structural work. In an extreme case where a longer column is required, use Euler's formula with $E = 13{,}000{,}000$ pounds per square inch or less.

Problems

1. Calculate the total safe load, by American Railway Formula, for 6-inch by 6-inch by 1-inch angle 10 feet long as a column.

Ans. 96,350 pounds.

2. Find the total safe load on a plate and channel column 20 feet long, made of two 10-inch 15-pound channels, placed 6 inches back to back, and two 12-inch by $\frac{1}{4}$-inch plates. *Ans.* 169,500 pounds.

3. Find the diameter of a solid circular steel column 5 feet long to carry a load of 60,000 pounds. *Ans.* 2.77 inches.

148. Straight-line Formulas for Square or Fixed Ends.—In applying straight-line formulas to columns with square or fixed ends, it is customary to *modify* the *constant* k and use the entire length of the column as l in the formula. The American Railway Engineering and Maintenance of Way Association uses the one constant (70) for all cases, treating the so-called fixed and square ends no better than hinged ends. This is good practice for bridges and similar structures. When a bridge post is riveted to the floor beam, experiments show that the deflection of the beam often produces a bending stress in the post which is equivalent to a large eccentricity. In pin-connected bridges, a slight difference in the length of the eye-bars which form the diagonals of the truss sometimes causes such concentration of stress in one side of the post that it is weaker in the plane of the pins than perpendicular to that plane. In buildings, where the floor beams are riveted to the posts, there is likely to be considerable eccentricity in the end posts. At intermediate posts with beams on both sides the eccentricity is less.*

The American railway formula may well be used for all structures built of structural steel, provided, of course, due allowance is made for live loads and impact in computing the total load.

* See paper by C. T. MORRIS, *Engineering News,* Nov. 2, 1911, page 53C.

The building laws of New York City require for structural-steel columns with square or fixed ends:

$$\frac{P}{A} = 15{,}200 - 58\frac{L}{r}. \tag{1}$$

If we regard the length of the sine curve as 0.8 of the total length of the column, we get from the American railway formula $0.8 \times 70 = 56$. If k is 70 for round ends, and the effect of square or fixed ends is sufficient to make a 10-foot column equal in strength to an 8-foot round-end column, the constant 58 may be taken as practically correct for the square and fixed ends, if the total length of the column is taken as L (see experiments, Table XIII).

For square-end timber columns of long-leaf yellow pine the New York Building Laws specify

Wood

$$\frac{P}{A} = 1{,}000 - 18\frac{L}{D}, \tag{2}$$

where D is the least transverse dimension. In a solid circular section the radius of gyration is one-fourth of the diameter, so that $18\frac{L}{D}$ is equivalent to $4.5\frac{L}{r}$. In Problem 4 of Article 145 the second term of the straight-line formula for yellow pine was found to be $6\frac{l}{r}$ for a column with round ends. If the ends are so fixed that l is a little less than $\frac{3L}{4}$, where L is the entire length of the square-end or fixed-end column and l is the length of the round-end column which will carry the same load, then $6\frac{l}{r}$ is practically equivalent to $18\frac{L}{D}$.

For a column of rectangular section $r = \frac{D}{\sqrt{12}}$ so that equation (2) is safer for a rectangular column than for a solid circular one.

For cast-iron columns the Syracuse Building Laws specify

$$\frac{P}{A} = 9{,}000 - 40\frac{l}{r}. \tag{3}$$

This is a conservative formula and is recommended where it is necessary to use cast-iron columns.

Some writers regard square-end and "fixed-end" columns as being really fixed at the ends, and consider pin-end columns as

equivalent to round ends. They give k for the square-end and "fixed-end" columns as one-half as great as for pin ends, and for pin-and-square columns they use for k a value seven-tenths or two-thirds that for round ends. As no columns are perfectly fixed and as pin-end columns are partly fixed by the friction, this difference is entirely too great, and such constants are dangerous.

When it is necessary to design a column to carry a given load, if the section is circular or rectangular, the diameter or breadth may be calculated directly from the formula. Where rolled shapes are used as columns and where the columns are built up of shapes and plates the desired section must be found by the method of trial and error. Where the handbooks give tables of the safe loads on columns calculated by other formulas, these may be used to find the approximate section.

Example

Find the I-beam as a column 10 feet long to carry a load of 40,000 pounds in accordance with the formula of the New York Building Laws.

Using the table in Cambria it is found that an 8-inch 18-pound I-beam is calculated to carry a load of 43,000 pounds by the formula there used. Applying the New York formula:

$$\frac{P}{5.33} = 15{,}200 - \frac{58 \times 120}{0.84} = 6{,}914 \text{ pounds},$$

$$P = 36{,}850 \text{ pounds}.$$

Trying the 9-inch 21-pound I-beam, it is found to be able to carry safely over 47,000 pounds.

Problems

1. Find the total safe load by New York Building Laws on a 6-inch by 8-inch yellow-pine post 12 feet long. *Ans.* 27,264 pounds.

2. A yellow-pine post 15 feet long to carry 60,000 pounds is of rectangular section and is 10 inches one way. Find the other dimension by New York formula. *Ans.* 9.24 inches.

3. Solve Problem 2 for a load of 90,000 pounds. *Ans.* 13.31 inches.

4. Select an I-beam to be used as a column 15 feet long to carry a load of 70,000 pounds in accordance with the New York formula. *Ans.* 12-inch 45-pound I-beam.

5. Select a plate and channel column 25 feet long to carry a load of 120,000 pounds.

149. Gordon's or Rankine's Formulas.—While the straight-line formulas have recently come into general use among American engineers, on account of the ease of application and the fact that they agree as well with the results of tests and with exact

theory as the more complicated expressions, another type of working formula had the preference until a few years ago, and is still the favorite with British engineers. This type is called Gordon's or Rankine's formula. It is an empirical formula which gives the unit load equal to the ultimate strength for a short block and which approaches Euler's curve for a very long column. It is:

$$\frac{P}{A} = \frac{s_u}{1 + q\left(\frac{l}{r}\right)^2}, \qquad \text{Formula XXX.}$$

where s_u is the ultimate strength in compression in the case of a short block and q is a coefficient, the value of which may be determined experimentally. To use the formula with any given factor of safety, it is merely necessary to divide the numerator by the factor; that is, to use the allowable unit stress instead of the ultimate strength.

Example

The Philadelphia Building Laws specify for medium steel columns in buildings,

$$\frac{P}{A} = \frac{16{,}250}{1 + \frac{L^2}{11{,}000\, r^2}}.$$

Find the total safe load on a 4-inch solid steel cylinder 10 feet in length as a column.

The total length is 120 inches and the radius of gyration is 1 inch.

$$\frac{P}{A} = \frac{16{,}250}{1 + \frac{14{,}400}{11{,}000}} = \frac{16{,}250}{1 + 1.309} = 7{,}038.$$

$$P = 7{,}038 \times 12.566 = 88{,}400 \text{ pounds.}$$

Problems

1. Find the total safe load by Philadelphia formula on a 6-inch by 6-inch by 1-inch angle of medium steel as a column 10 feet long.

Ans. 90,600 pounds.

2. Find the total safe load by Philadelphia formula on a 12-inch 40-pound I-beam as a column 15 feet long.

Ans. $\frac{P}{A}$ = 3,870 pounds per square inch; P = 45,500 pounds.

3. Find the total safe load by Philadelphia formula on a post made of two 10-inch 20-pound channels placed 6 inches back to back and latticed together, for lengths of 15 feet and 20 feet.

The values of the total safe load on columns given in Cambria Steel are calculated from the formula:

$$\frac{P}{A} = \frac{12{,}500}{1 + \dfrac{l^2}{36{,}000\, r^2}}. \tag{2}$$

for square-end columns. For pin-end columns $\frac{1}{18{,}000}$ is taken as the value of q. These are Rankine's values and were based on a limited number of tests of relatively short columns. Both are too small and err on the side of danger. On the other hand, the allowable unit stress of 12,500 used as the numerator is more conservative than the figures generally used, so that the tables give good values for all columns except relatively long ones. The figures for the long columns are safe, but the factor of safety is less than for the shorter ones.

Problems

Find the total safe load on the following columns by equation (2) and compare the results with the Cambria tables.

4. A 10-inch 35-pound I-beam 12 feet long.

5. A 15-inch 60-pound I-beam 15 feet long.

6. A plate and angle column, 20 feet long, made of one 12-inch by ½-inch plate and four 4-inch by 3-inch by ½-inch angles.

150. Ritter's Rational Constant for Rankine's Formula.—While the constant q was originally derived from a few tests of columns, it may be obtained from the constants of the materials. We know from experiments and theory that Euler's formula gives the ultimate load when the load is exactly central, the ends either perfectly free to turn or absolutely fixed, and the value of $\frac{l}{r}$ so great that the computed $\frac{P}{A}$ is below the true elastic limit of the material. Any curve which is to be used with all lengths must coincide with Euler's curve when $\frac{l}{r}$ becomes indefinitely large, and must also pass through the point s_u when $\frac{l}{r} = 0$.

We see that when $\frac{l}{r}$ equals zero, $\frac{P}{A}$ equals s_u; Rankine's formula satisfies the second of the above-mentioned conditions. To make it satisfy the first condition, we must find some value of

q which will make $\frac{P}{A}$ the same in Rankine's and Euler's formulas for large values of $\frac{l}{r}$:

$$\frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2} = \frac{s_u}{1 + q\left(\frac{l}{r}\right)^2}. \tag{1}$$

For large values of $\frac{l}{r}$ the second term in the denominator of Rankine's formula is so large relatively that the first term (unity) may be dropped. Then

$$\frac{\pi^2 E}{\left(\frac{l}{r}\right)^2} = \frac{s_u}{q\left(\frac{l}{r}\right)^2} \tag{2}$$

$$q = \frac{s_u}{\pi^2 E} \tag{3}$$

This value of q is *Ritter's rational constant.*

Problems

1. Find the value of q for steel for which the modulus of elasticity is 29,000,000 pounds per square inch, and the ultimate compressive strength is 36,000 pounds per square inch. *Ans.* $q = \frac{1}{7{,}950}$

2. Taking q as $\frac{1}{8{,}000}$ and the ultimate strength as 36,000 pounds per square inch, find the ultimate unit load, in pounds per square inch, for values of the slenderness ratio at intervals of 40 from 40 to 200.

Ans.

$\frac{l}{r}$	40	80	120	160	200
$\frac{P}{A}$	30,000	20,000	12,857	8,571	6,000

3. Solve Problem 2 by Euler's formula taking $\pi^2 E = 288{,}000{,}000$, which corresponds with $q = \frac{1}{8{,}000}$ when $s_u = 36{,}000$ pounds per square inch.

4. Using the constants of Problem 2 find the total safe load on a solid circular column 4 inches in diameter and 50 inches long with a factor of safety of 2.25.

Curve I of Fig. 177 is drawn from Rankine's formula for steel for which the working unit stress is 16,000 pounds per square inch and the working modulus of elasticity is a little less than 13,000,000 pounds per square inch, making Ritter's constant $\frac{1}{8{,}000}$. This is practically the same as Problem 4 with a factor

of safety of 2.25. Curve II is from the secant formula with $\frac{ec}{r^2} = 0.1$. For the slenderness ratios which are chiefly used in structural columns the results of Rankine's formula with Ritter's constants are on the safe side, but are not as close to the true values as those given by the straight-line formulas.

Curve III is calculated from $q = \frac{1}{18,000}$, which is Rankine's constant for round-end steel columns. For relatively short

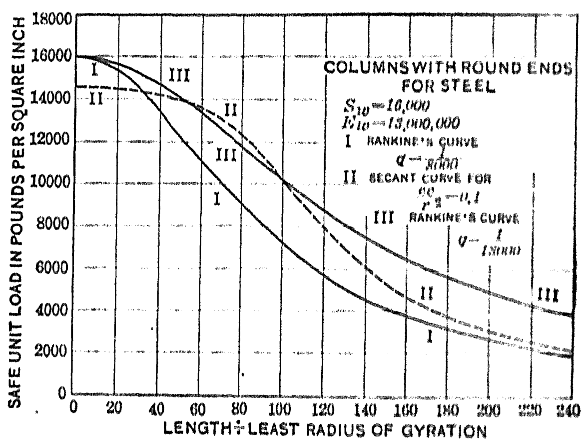

FIG. 177.—Curves for steel columns.

columns the results are approximately correct, but for long columns there is a large error on the side of danger.

Since the only advantage possessed by Rankine's formula over a straight-line formula is the fact that the former may be used with columns of any slenderness ratio, it follows that if it is used at all it should be taken with Ritter's constant so that the errors would always be on the safe side, except with very short columns with considerable eccentricity.

151. Rankine's Formula for Square-end and Fixed-end Columns.—In adapting Rankine's formula to square-end and fixed-end columns the entire length L is employed in the formula and q is modified. If the column were absolutely fixed at the ends, the q of Ritter's formula would be divided by 4, so that a q of $\frac{1}{8,000}$ for round ends would be $\frac{1}{32,000}$ for square ends. In actual columns, since the ends are not perfectly fixed, some ratio

less than 4 must be used to multiply the denominator of the fraction. (Rankine used 2 as the ratio.) Suppose that l be taken as 0.8 L, then $l^2 = 0.64\ L^2$, and $\frac{l^2}{7{,}950\ r^2}$ from Problem 1 of Article 150 becomes $\frac{L^2}{12{,}409\ r^2}$. The Philadelphia formula for medium steel uses $q = \frac{1}{11{,}000}$ for medium steel, which agrees fairly well with the above figure and is on the safe side.

For mild steel the Philadelphia Building Laws specify

$$\frac{P}{A} = \frac{14{,}500}{1 + \frac{L^2}{13{,}500\ r^2}}.$$

Problems

1. Find the total safe load by New York and by Philadelphia formula for a 15-inch 42-pound I-beam of medium steel as a column 12 feet long.
2. Solve Problem 1 for a length of 20 feet by Philadelphia formula and by Euler's formula taking $E_w = 13{,}000{,}000$ pounds per square inch, and l as 0.8 of the total length.

In concrete columns it is customary to specify an allowable unit load in pounds per square inch and a maximum ratio of length to diameter or minimum breadth. Most cities specify a unit compressive stress of 500 pounds per square inch and the maximum $\frac{L}{D}$ as 15. The report of the Joint Committee on Concrete and Reinforced Concrete gives 450 pounds per square inch for 1 : 2 : 4 concrete having an ultimate strength of 2,000 pounds per square inch.

Problems

3. Concrete columns 15 feet long are designed to carry a floor load of 400 pounds per square foot. If the columns are circular and are spaced 12 feet apart both ways what should be their diameter?
4. Solve Problem 3 if the columns are spaced 16 feet apart both ways.

152. General Conclusions.—The calculation of columns is not as satisfactory as that of beams. This is due to two reasons: the location of the load, and the relative freedom of the ends. In a beam, the location of the load is known with a large relative accuracy. A 1-inch displacement of the load in a horizontal beam 10 feet long produces a very small effect upon the unit stress; an equal displacement of the load at the end of a block 6

inches square will *double* the maximum stress on one face. Again, we generally deal with beams entirely free to turn at the supports or with cantilevers which are entirely free to move and turn at one end and which are perfectly fixed at the other, so far as concerns the moment arms. The results which we get in calculating beams are correct inside the true elastic limit and approximately true beyond that limit. If we take a column perfectly free to turn at both ends and know the position of the load with the same *relative* certainty as in the case of the beam supported at the ends, we may calculate the unit stress with the aid of Formula XXV as accurately as we can compute it in the beam by the use of Formula XV. There is this apparent difference: in the beam the unit stress varies as the load; in the column it increases more rapidly. Again, a column fixed at one end and free at the other (case II, Fig. 168) can be calculated with the same accuracy as a cantilever with one end free, provided the load is located with the same relative accuracy and the end is so well fixed that the *relative* change in moment due to change in tangent at the "fixed end" is the same in both cases. The change in moment due to change in the tangent at the ends is proportional to the rate of change in the cosine of a small angle in the case of a beam and to the rate of change of the sine of a small angle in the column. The loads being much greater in a column than in a beam of the same section, the effect of friction in partially fixing the ends is greater in the column.

Beams fixed at both ends or fixed at one end and supported at the other are indefinite, because it is not possible to fix the beam perfectly so that it will not turn, or support it so that it will not move. For these reasons the calculation of the unit stresses in relatively stiff beams of these kinds is always open to question. The same is true of columns fixed at both ends, or fixed at one end with a hinge connection at the other.

The error in the case of the fixed column is relatively greater than in a fixed beam, for a change in the slope of the tangent at the ends of the column makes a relatively larger change in the bending moment.

Euler's formula gives the ultimate load which will cause a column with practically no eccentricity to deflect without limit. Unless the slenderness ratio is large the column will fail by crushing before this load is reached. *If the value of* $\frac{P}{A}$, *calculated by*

Euler's formula, is greater than the elastic limit of the material, they must be discarded and the calculation repeated with a formula which fits shorter columns. It is best to limit the use of Euler's formula to slenderness ratios where it gives values of $\frac{P}{A}$ which, after division by the factor of safety, are less than one-third of the allowable unit compressive stress of the material.

For shorter columns draw a straight line through the proportional elastic limit of the material and tangent to Euler's curve and divide by the factor of safety, or construct an Euler's curve by means of a working modulus of elasticity obtained by dividing E by the factor of safety and draw a line tangent thereto through an intercept on the Y axis of which the ordinate is the allowable unit compressive stress.

With hinged-end columns the l of these formulas is the entire length between hinges. With case II, l is twice the length of the column. With fixed ends or with pin-and-square ends, it is also safest to take l as the entire length of the column. If any allowance is made, it should not be as great as that of the ideal cases, which are never met in practice. The amount of allowance depends upon the relative dimensions of the column and the beams to which it is attached and the method of attachment.

The effect of eccentricity is taken into account by using a limiting stress for short columns, as in the case of the American railway formula, and by the use of a large factor of safety (well called a factor of ignorance) to take care of any uncertainties in this respect. (The real factor of safety in many columns which are standing is probably much less than figured by the designer.)

Rankine's formula is used by some engineers. With Ritter's constants it is always safe—unnecessarily safe for columns of moderate length. With Rankine's constants it should not be used for long columns.

If the eccentricity of the load were sufficiently well known, the secant formulas of Article 136 are strictly correct for round-end columns, provided the stress is below the elastic limit. A set of curves like those of Fig. 174 may be drawn and employed in the calculations to save labor.

153. Failure of Beams Due to Flexure of the Compression Flange.—The Compression Flange of a beam acts as a column

and may fail by lateral deflection. The unit stress in the extreme outer compression fibers is the unit load $\frac{P}{A}$ of the column theory. The American Railway Engineering and Maintenance of Way Association specifications give the formula,

$$\frac{P}{A} = 16{,}000 - 200\,\frac{l}{b}, \tag{1}$$

as the maximum value of the compressive unit stress in any beam or girder when the compression flange is made of angles only or of angles and flat plates. In this formula b is the breadth of the flange and l is the distance between lateral supports. If the flange be regarded as rectangular $b^2 = 12\,r^2$, $b = 3.46\,r$ and equation (1) is equivalent to:

$$\frac{P}{A} = 16{,}000 - 58\,\frac{l}{r}. \tag{2}$$

This coefficient of 58 in the straight-line formula is permissible since only the extreme outer fibers at the dangerous section reach the maximum unit stress. In the one case of a beam under constant moment the maximum unit stress occurs in the outer fibers of the entire length, and a larger coefficient should be employed.*

The formula recommended by the American Bridge Co. is

$$\frac{P}{A} = 19{,}000 - 300\,\frac{l}{b} \tag{3}$$

with a maximum of 16,000 pounds per square inch.

Problems

1. Calculate by equation (1) the maximum distance between lateral supports for a 12-inch 31.5-pound I-beam if the maximum unit bending stress is 14,000 pounds per square inch. *Ans.* 50 inches.

2. What is the maximum allowable bending stress in an 18-inch 55-pound I-beam 12 feet long with no lateral supports for the compression flange. Solve by equation (1). *Ans.* 11,200 pounds per square inch.

3. Solve Problem 2 by the American Bridge Co. formula.
Ans. 11,800 pounds per square inch.

* A full discussion of this subject is to be found in *Bulletin* No. 68 of The Illinois University Engineering Experiment Station, by Prof. Herbert F. Moore.

See also paper by R. Fleming in *Engineering News*, April 6, 1916, and paper by Henry Kercher in *Engineering News*, May 4, 1916.

The table in Cambria in the article entitled "Lateral Strength of Beams without Lateral Support" was derived from the Rankine formula for fixed-end columns.

$$\frac{P}{A} = \frac{18,000}{1 + \frac{l^2}{36,000\, r^2}} = \frac{18,000}{1 + \frac{l^2}{3,000\, b^2}} \tag{4}$$

by substituting b^2 for $12\, r^2$. The numerator is taken at 18,000 pounds per square inch instead of the usual 16,000 pounds per square inch on account of the fact that only a part of the flange is subjected to the maximum unit stress.

Problems

4. Find the total safe load uniformly distributed on a 24-inch 80-pound I-beam 20 feet long supported at the ends with no lateral supports. Use equation (4).

Ans. Maximum unit bending stress = 12,930 pounds per square inch: total load = 74,951 pounds.

5. Solve Problem 4 by equation (1).

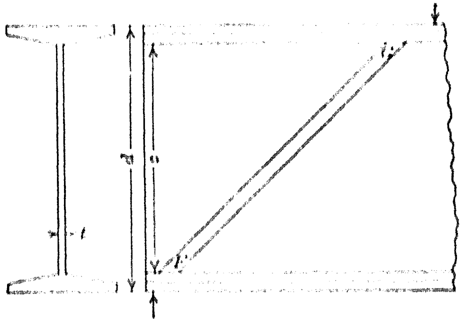

FIG. 178.—Web of I-beam as a column.

154. Failure Due to Buckling of the Web. It was shown in Article 31 that vertical shear produces compressive stress which is a maximum at 45 degrees with the vertical, where its intensity is equal to the unit vertical or horizontal shearing stress. The web of an I-beam subjected to vertical shear may be regarded as made up of a series of parallel columns with fixed ends. In Fig. 178 *FG* represents one such column. The thickness of this column is t the thickness of the web; and its length is $\sqrt{2}c$, where c is the distance between flanges (represented by l in the Cambria diagrams). It is customary to find the mean vertical shear in the web of an I-beam by dividing the total vertical shear by the area td, where t is the thickness of the web and d is the entire

depth of the beam. Since unit compressive stress is equal to the unit shearing stress,

$$\frac{P}{A} = \frac{V}{td}, \tag{1}$$

where $\frac{P}{A}$ is the unit load in the column formula and V is the total vertical shear. To find the safe value for V, it is necessary only to solve $\frac{P}{A}$ by any column formula, remembering that $r^2 = \frac{t^2}{12}$ for a rectangular section of thickness t. Using first Rankine's formula with 12,000 as the numerator and q equal to $\frac{1}{36,000}$ in order to compare with Cambria, we get:

$$\frac{P}{A} = \frac{12,000}{1 + \frac{l^2}{36,000\, r^2}} \tag{2}$$

$$\frac{P}{A} = \frac{12,000}{1 + \frac{1}{1,500}\left(\frac{c}{t}\right)^2} \tag{3}$$

Problems

1. Find the maximum value of the unit shear, the total vertical shear, and the total load uniformly distributed, on a 12-inch 31.5-pound I-beam, by means of the above formula.

Ans. 7,488 pounds per square inch, 31,450 pounds, and 62,900 pounds.

2. Solve Problem 1 for a 15-inch 42-pound I-beam. Compare results with Cambria under "Maximum Loads of I-beams and Channels Due to Crippling the Web."

3. If the allowable unit stress due to bending is 16,000 pounds per square inch, what is the minimum length for which the full bending stress may be developed by a uniformly distributed load without producing excessive buckling stresses in a 12-inch 31.5-pound I-beam?

4. Solve Problem 3 for a 20-inch 65-pound I-beam for the maximum load and minimum span without crippling the web if the load is concentrated at the middle.

5. Solve Problem 1 by the American railway formula.

Ans. Total load, 47,780 pounds.

6. Solve Problem 1 by the New York Building Laws.

CHAPTER XV

RESILIENCE IN BENDING AND SHEAR

155. Resilience in Beams.—In Article 12 it was shown that elastic resilience per cubic inch is $\frac{s^2}{2E}$, and that the total energy is that quantity multiplied by the volume. In beams the unit stress varies as the distance from the neutral surface and also varies with the moment at the section. The total elastic energy may be determined in one of two ways: the total work done by the *external forces* may be calculated, or an expression for the *internal stress s* may be derived and this expression integrated over the entire volume.

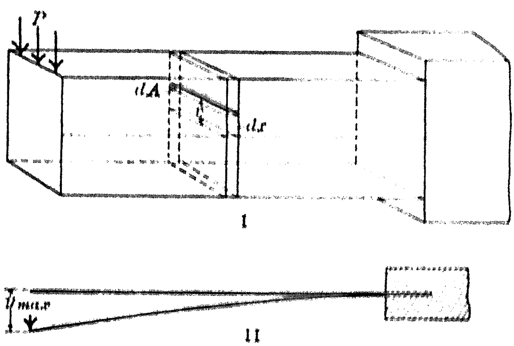

FIG. 179.—Work of deflection.

By external work, if a load P causes a deflection y_{max} at its point of application, the work is $\frac{Py_{max}}{2}$. In a cantilever with a single concentrated load, Fig. 179,

$$y_{max} = \frac{Pl^3}{3\,EI},$$

$$\text{External work} = \frac{P^2l^3}{6\,EI} \tag{1}$$

In a beam supported at the ends with a load at the middle,

$$\text{External work} = \frac{P}{2} \times \frac{Pl^3}{48\,EI} = \frac{P^2l^3}{96\,EI} \tag{2}$$

Where the load is uniformly distributed, w pounds per unit length, each increment of load wdx on a length dx does work amounting to $\frac{wydx}{2}$, where y is the deflection of the particular part of the beam on which the increment rests. In Fig. 180, II one increment wdx is deflected a distance y_1, another, y_2, etc. The different values of y are determined from the equation of the

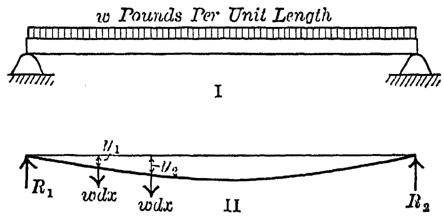

FIG. 180.—External work.

elastic line. The total work is the sum of these increments of work.

$$\text{Total work} = \frac{w}{2}\int ydx \qquad (3)$$

with the ends of the beam as the limits.

156. Expression for Internal Work.—In a beam the unit stress at a distance v from the neutral axis is $\frac{Mv}{I}$. In Fig. 179, I, there is an element of volume of cross-section dA and length dx at a distance v from the neutral axis. The energy dU in this element of volume $dAdx$ is

$$dU = \frac{s^2}{2E}\,dAdx = \frac{M^2v^2}{2\,EI^2}\,dAdx. \qquad (1)$$

$$\text{Total work in beam} = \int\int \frac{M^2}{2\,EI^2}\,v^2dAdx. \qquad (2)$$

Integrating first with respect to v gives the work done upon the volume of length dx between two vertical planes. Throughout this volume x, M, and I are constant. The integral of v^2dA across the beam from the bottom to the top is I.

$$\text{Work} = \int \frac{M^2}{2\,EI}\,dx \qquad (3)$$

Equation (3) may be used to calculate the internal work in any beam. Before integrating M and I must be expressed as functions of x unless they are constant.

157. Beam with Constant Moment.—For the case of a beam with constant moment and constant moment of inertia,

$$U = \frac{M^2}{2\,EI}\int dx = \frac{M^2}{2\,EI}[x]_0^l = \frac{M^2 l}{2\,EI}.$$

To find the work in terms of the maximum fiber stress, if the neutral axis passes through the center of the section so that the distance to the outer fibers is $\frac{d}{2}$, $M = \frac{2\,SI}{d}$, which substituted in (1) gives

$$U = \frac{4\,S^2I^2l}{2\,EId^2} = \frac{2\,S^2Il}{Ed^2}. \tag{2}$$

For a rectangular section

$$U = \frac{2\,S^2bd^3l}{12\,Ed^2} = \frac{S^2bdl}{6\,E} = \frac{S^2}{6\,E} \times \text{volume}. \tag{3}$$

The average energy per unit of volume is $\frac{S^2}{6\,E}$, which is one-third as great as that in a block subjected to a uniform stress S.

Problems

1. Find the average energy per unit volume in a solid circular section subjected to a uniform bending moment. *Ans.* $\frac{S^2}{8\,E}$.

2. A steel bar 2 inches wide and ½ inch thick is 8 feet long and rests on two supports 6 feet apart and carries two equal loads on the ends. If E is 30,000,000 pounds per square inch, what is the total elastic energy in the part between the supports when each load on the ends is 100 pounds? *Ans.* 82.9 inch-pounds.

158. Beam with Uniformly Distributed Load.—*For a cantilever with uniformly distributed load* with the origin at the free end, the moment is $\frac{wx^2}{2}$ and

$$U = \frac{w^2}{8\,E}\int_0^l \frac{x^4}{I}\,dx. \tag{1}$$

When I is constant

$$U = \frac{w^2l^5}{40\,EI} = \frac{W^2l^3}{40\,EI}. \tag{2}$$

For a section which is symmetrical with respect to the neutral axis

$$\frac{Wl}{2} = \frac{2\,SI}{d},$$

$$U = \frac{4\,S^2Il}{10\,Ed^2}. \tag{3}$$

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For a rectangular section where $I = \frac{bd^3}{12}$,

$$\text{Total work} = \frac{S^2bdl}{30\,E} = \frac{S^2}{30\,E} \times \text{volume}. \qquad (4)$$

The total energy in a cantilever of rectangular section with uniformly distributed load is one-fifteenth as much as that in a block of the same volume with uniform compressive stress throughout.

To find the total elastic energy by the method of external work, the equation of the elastic line from equation (9) of Article 77 is

$$y = \frac{w}{24\,EI}(x^4 - 4\,l^3x + 3\,l^4),$$

if deflection downward is regarded as positive. Substituting in equation (3) of Article 155 and taking the load w as positive downward,

$$\text{Total work} = \frac{w}{2}\int y\,dx = \frac{w^2}{48\,EI}\int_0^l (x^4 - 4\,l^3x + 3\,l^4)\,dx \qquad (5)$$

$$\text{Total work} = \frac{w^2}{48\,EI}\left(\frac{l^5}{5} - 2\,l^5 + 3\,l^5\right) = \frac{w^2l^5}{40\,EI} \qquad (6)$$

For a *beam supported at the ends with uniformly distributed load*, $M = \frac{wlx}{2} - \frac{wx^2}{2}$.

$$U = \frac{w^2}{8\,EI}\int (l^2x^2 - 2\,lx^3 + x^4)\,dx, \qquad (7)$$

$$U = \frac{w^2}{8\,EI}\left[\frac{l^2x^3}{3} - \frac{lx^4}{2} + \frac{x^5}{5}\right]_0^l = \frac{w^2l^5}{240\,EI} \qquad (8)$$

By external work taking y positive downward,

$$\int \frac{wy}{2}\,dx = \frac{w^2}{2\,EI}\int\left(-\frac{lx^3}{12} + \frac{x^4}{24} + \frac{l^3x}{24}\right)dx, \qquad (9)$$

$$\text{Work} = \frac{w^2}{2\,EI}\left[-\frac{lx^4}{48} + \frac{x^5}{120} + \frac{l^3x^2}{48}\right]_0^l = \frac{w^2l^5}{240\,EI} \qquad (10)$$

Problem

Find the work per unit volume in terms of the maximum unit stress in a beam of rectangular section which is supported at the ends and uniformly loaded.

Ans. $\frac{4\,S^2}{45\,E}$.

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159. Beam with Single Concentrated Load.—For a cantilever with a load on the free end, $M = -Px$. When the section is constant

$$U = \frac{P^2}{2\,EI}\int x^2 dx = \frac{P^2}{6\,EI}\Big[x^3\Big]_0^l = \frac{P^2 l^3}{6\,EI}. \tag{1}$$

For a beam supported at the ends with a load P at a distance a from one end and at a distance b from the other, the reaction at the end of the length a is $\frac{Pb}{l}$ and the moment in this length is $\frac{Pbx}{l}$. The work in this part of the beam is,

$$U = \frac{P^2 b^2}{2\,EIl^2}\int_0^a x^2 dx = \frac{P^2 b^2 a^3}{6\,EIl^2}. \tag{2}$$

Similarly in the length b,

$$\text{Work} = \frac{P^2 a^2 b^3}{6\,EIl^2}. \tag{3}$$

For the entire length,

$$\text{Total work} = \frac{P^2 a^2 b^2\,(a+b)}{6\,EIl^2} = \frac{P^2 a^2 b^2}{6\,EIl}. \tag{4}$$

Problems

✓**1.** Find the elastic energy per unit volume in a cantilever of rectangular section in terms of the maximum unit stress. [illegible] Cantilever [illegible] loaded [illegible] *Ans.* $\frac{S^2}{18\,E}$.

2. Find the elastic energy per unit volume in a hollow cylindrical cantilever in terms of the maximum unit stress.

Ans. $U = \frac{S^2(R_2^2 + R_1^2)}{24\,R_2^2 E}$, where R_2 is the outside radius and R_1 is the inside radius.

160. Beam of Variable Section.—In a beam of variable section the moment of inertia is a variable. In any beam at a distance v from the neutral axis $s = kv$ where k is a constant for that section.

$$\text{Work per unit volume} = \frac{k^2 v^2}{2\,E}. \tag{1}$$

For a rectangular section of breadth b the element of volume is $bdvdx$. For the integration with respect to v the terms k and b are constant.

$$U = \frac{1}{2\,E}\int bk^2 \int_{-\frac{d}{2}}^{\frac{d}{2}} (v^2 dv)\,dx = \frac{1}{24\,E}\int bk^2 d^3 dx \tag{2}$$

(160):—

$$\mathcal{U} = \int \frac{M^2}{2EI}dx = \int \frac{S^2 I^2}{2EIc^2}dx = \int \frac{S^2 b d^3 \times 4}{12 \times 2Ed^2}dx = \frac{S^2}{6E}\int bd\cdot dx$$

$$= \frac{S^2}{6E} \times \text{volume}$$

This holds good for any [illegible] uniform [illegible] is constant moment

The maximum fiber stress $S = \frac{kd}{2}$, substituting which,

$$U = \frac{1}{6\,E}\int S^2 bd\,dx. \tag{3}$$

If the beam be so designed that S is constant for all sections

$$U = \frac{S^2}{6\,E}\int bd\,dx = \frac{S^2}{6\,E} \times \text{volume}. \tag{4}$$

The elastic energy per unit volume in a beam of constant strength and rectangular section is $\frac{S^2}{6\,E}$, which is one-third as great as that in a block subjected to uniform compressive stress equal to S.

161. Deflection Calculation by Internal Work.—The work done in a beam affords a method of finding the deflection under a concentrated load. In the case of a cantilever with a load on the end, the average force is $\frac{P}{2}$ and the work of deflection is $\frac{Py_{\max}}{2}$. This is equal to the internal work (equation (1) of the preceding article).

$$\frac{Py_{\max}}{2} = \frac{P^2 l^3}{6\,EI}, \tag{1}$$

$$y_{\max} = \frac{Pl_3}{3\,EI}. \tag{2}$$

For a beam supported at the ends with a concentrated load at a distance a from one end, the deflection under the load is given by

$$\frac{Py}{2} = \frac{P^2a^2b^2}{6\,EIl}, \quad y = \frac{Pa^2b^2}{3\,EIl}. \tag{3}$$

Problems

1. A 4-inch by 6-inch wooden beam 20 feet long is supported 5 feet from each end and carries a load of 200 pounds on each end. Find the deflection at the ends, if E is 1,000,000 pounds per square inch. *Ans.* 0.8 inch.

2. A 6-inch by 2-inch beam 20 feet long is supported 5 feet from the left end and held down at the left end. A load of 20 pounds is placed on the right end. Find the deflection if E = 1,200,000. *Ans.* 10.8 inches.

For a *cantilever beam of constant strength and rectangular section* when the depth is constant the breadth varies as the distance from the free end so that the plan is triangular. Volume $= \frac{BDl}{2}$, where B and D are the maximum breadth and depth.

$$\text{Total work} = \frac{S^2}{6\,E} \times \frac{BDl}{2} = \frac{Py_{\max}}{2}, \tag{4}$$

Substituting

$$S = \frac{PlD}{2\,I_{\max}},$$

$$\text{Total work} = \frac{P^2BD^3l^3}{48\,EI^2_{\max}} = \frac{Py_{\max}}{2}. \tag{5}$$

Since

$$I_{\max} = \frac{BD^3}{12},$$

$$\frac{P^2l^3}{4\,EI_{\max}} = \frac{Py_{\max}}{2}, \tag{6}$$

$$y_{\max} = \frac{Pl^3}{2\,EI_{\max}}.$$

(Compare with Article 121.)

In a rectangular cantilever beam of constant strength and constant breadth with a load on the end, the depth is given by the equation of the parabola

$$d^2 = \frac{6\,Px}{SB}.$$

The area is $\frac{2\,Dl}{3}$ and the volume is $\frac{2\,BDl}{3}\cdot$ Making the same substitutions as in the preceding case,

$$y_{\max} = \frac{2\,Pl^3}{3\,EI_{\max}}. \tag{7}$$

Problem

3. Find the deflection at the end of a cantilever of constant strength and square section due to a load on the end. *Ans.* $y_{\max} = \frac{3\,Pl^3}{5\,EI_{\max}}.$

162. Internal Work of Shear in a Shaft.—The unit shearing stress s_s produces a deformation of $\frac{s_s}{E_s}$ in planes at unit distance apart. The work of shear is the product of half the unit stress by the total deformation,

$$\text{Work per unit volume} = \frac{s_s}{2} \times \frac{s_s}{E_s} = \frac{s_s^2}{2\,E_s}. \tag{1}$$

In a solid circular shaft at a distance r from the axis, the unit shearing stress is kr and

$$\text{Energy per unit volume} = \frac{k^2r^2}{2\,E_s}. \tag{2}$$

The element of volume of length l is $2\ \pi r l\ dr$ and

$$\text{Total energy} = \frac{\pi k^2 l}{E_s}\int_0^a r^3 dr = \frac{\pi l k^2 a^4}{4\ E_s}. \qquad (3)$$

where a is the radius of the shaft. The maximum unit shearing stress in the outer surface is $S_s = ka$ and

$$\text{Total energy of shear} = \frac{S_s^2}{4\ E_s}\pi a^2 l = \frac{S_s^2}{4\ E_s} \times \text{volume}. \qquad (4)$$

(Compare Article 45 where the energy of shear in a shaft is calculated by external work.)

The modulus of elasticity in shear is about two-fifths as great as in tension and compression, so that for equal values of the unit stress the total energy of a rod in torsion is one-fourth greater than that of the same rod in tension. However, since the elastic limit of steel and other similar materials in shear is less than in tension, the total energy which may be stored is about the same in both cases.

163. Work of Shear in a Rectangular Beam. In a beam of rectangular section of breadth b and depth d subjected to a vertical shear V

$$s_s = \frac{V}{Ib}\int bv\,dv = \frac{V}{I}\left[\frac{v^2}{2}\right]_v^{\frac{d}{2}} = \frac{V}{8\ I}(d^2 - 4\ v^2); \qquad (1)$$

$$\frac{s_s^2}{2\ E_s} = \frac{V^2}{128\ E_s I^2}(d^4 - 8\ d^2v^2 + 16\ v^4). \qquad (2)$$

Multiplying by the element of volume $b\ dv\ dx$ and integrating with respect to v with limits $-\frac{d}{2}$ and $\frac{d}{2}$,

$$U = \int \frac{V^2bd^5}{128\ E_s I^2}\left(1 - \frac{2}{3} + \frac{1}{5}\right) dx = \int \frac{3\ V^2}{5\ E_s bd}\,dx. \qquad (3)$$

When V is constant, the last term of (3) for a beam of constant section becomes $U = \frac{3\ V^2 l}{5\ E_s bd}$. In the case of a cantilever beam with a load on the free end, $V = -P$ and

$$U = \frac{3\ P^2 l}{5\ E_s bd}. \qquad (4)$$

To find the deflection due to shear at the end of a rectangular cantilever with a load on the end,

$$\frac{Py}{2} = \frac{3\,P^2l}{5\,E_sbd},$$

$$y = \frac{1.2\,Pl}{E_sbd} = \frac{1.2\,Vl}{E_sbd} = \frac{1.2\,s_s'l}{E_s}, \tag{5}$$

where s_s' is the average unit shearing stress.

The same relation holds for a beam supported at the ends with a concentrated load at the middle.

164. Sections of Maximum Resilience.—To obtain the maximum resilience per unit volume, the stress in all portions of the solid should be the maximum allowable unit stress. This condition of maximum efficiency can only be secured when the material is used in direct tension or compression, which is not practicable in the case of springs on account of the small displacement and the large force required, except in the case of soft rubber.

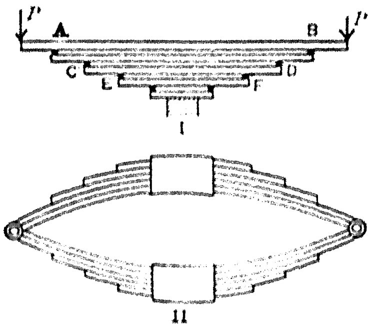


FIG. 181. Leaf spring.

In bending and torsion, only the outer fibers reach the maximum unit stress so that the energy per unit volume is always less than $\frac{S^2}{2\,E}$. In a rectangular beam of constant strength or in a rectangular beam of uniform section subjected to a constant moment, the energy per unit volume is $\frac{S^2}{6\,E}$, and the energy in such a beam used as a spring is one-third as great as would be possible if it were subjected to the maximum stress throughout, but it is three times as great as that of a beam of uniform section supported at the ends and loaded at the middle or used as a cantilever with a load on the end.

The common leaf springs (Fig. 181, II) are beams of constant strength made up of separate parts or leaves, each of which is subjected to constant bending moment throughout most of its length. In Fig. 181, I, the leaves are shown straight with each leaf resting on a pair of supports on the ends of the leaf below.

In the upper leaf the moment is constant from A to B, and the energy per unit volume in that portion is $\frac{S^2}{6E}$. The overhanging parts act as cantilevers loaded at the ends and the energy per unit volume is only one-third as great. In the actual leaf spring as shown in Fig. 181, II, the contact takes place over a considerable area and the stresses are modified by friction.

In an I-beam section a relatively large portion is in the flange, where the unit stress approximates the maximum, so that the energy per unit volume is greater than in a rectangular section.

In a solid circular section in torsion, the energy per unit volume was shown in Article 47 to be $\frac{S_s^2}{4E_s}$. In a hollow shaft of inside radius b and outside radius a,

$$\text{Total energy} = \frac{\pi k^2 l}{E_s}\int r^3 dr = \frac{\pi k^2 l}{4E_s}\Big[r^4\Big]_b^a \tag{1}$$

$$\text{Total energy} = \frac{\pi k^2 l\,(a^4 - b^4)}{4\,E_s}. \tag{2}$$

Dividing by $\pi\,(a^2 - b^2)l$, the energy per unit volume is,

$$U = \frac{k^2\,(a^2 + b^2)}{4\,E_s} = \frac{(a^2 + b^2)S_s^2}{4\,a^2 E_s} \tag{3}$$

since $ka = S_s$.

Problems

1. What is the energy per unit volume of a hollow cylinder in torsion if the inside diameter is three-fourths of the outside diameter? *Ans.* $\frac{25\,S_s^2}{64\,E_s}$.

2. What is the total elastic energy of torsion in a hollow steel rod 5 feet long, 1 inch outside diameter and ½ inch inside diameter, when the unit shearing stress is 80,000 pounds per square inch and E_s is 12,000,000 pounds per square inch. *Ans.* 5,890.5 inch-pounds.

3. A hollow rectangular beam is 6 inches by 8 inches outside and 4 inches by 4 inches inside. Find the energy per unit volume if the external moment is constant throughout the length. *Ans.* $\frac{11\,S^2}{48\,E}$.

165. Impact Stresses.—In the discussion of resilience it has been assumed that the load is applied gradually so that the average force is one-half the sum of the initial and final loads. If the initial load is zero and the final load P, the average load is $\frac{P}{2}$ and if the displacement under the load is y, the work is $\frac{Py}{2}$.

There are several ways of applying a load to meet these conditions. A load may be made up of a number of small parts and applied little at a time. For instance, a vessel may be hung from a beam or spring and gradually filled with water or sand. In testing large floors, bags of sand are added one at a time. In testing machines the loads are applied gradually by means of slowly moving screws.

Fig. 182 represents different ways of applying a single load. Fig. 182, I, shows a single spring. In Fig. 182, II, the same spring is shown with a mass W attached but with the entire

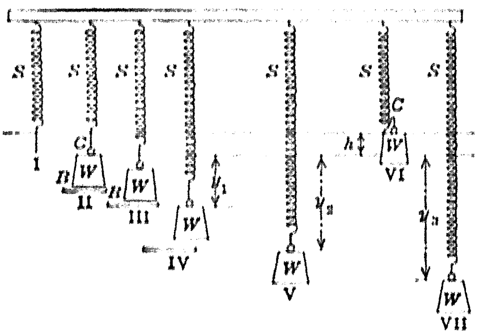


FIG. 182.—Effect of sudden loads and impact.

weight of W carried by a support B. In III the support has been lowered so that the tension in the spring carries part of the load. In IV the support has been entirely removed and the mass W is at rest with the spring stretched a distance y_1. When the elongation of the spring was one-half of y_1 the spring carried one-half the weight and the support the other half. As the elongation increased the load carried by the spring gradually increased and that on the support gradually decreased. The average load on the spring was $\frac{W}{2}$ and the average load on the support was the same. In moving the weight W the distance y_1 the total work done was Wy_1, half of this work was expended in stretching the spring and the remainder in assisting the motion of the support B.

If K is the force required to stretch the spring unit distance, the force required to stretch it the distance y_1 is $Ky_1 = W$, and the energy stored in the spring is $\frac{Ky_1^2}{2}$.

Ky is the [illegible] pulling the Spring.

If the support B is suddenly removed from W in the position II, the entire force of gravity is effective throughout the whole distance. At first the spring offers no resistance and the entire load goes to accelerate the mass (provided the mass of the spring is negligible). As it is stretched, the resistance of the spring increases. At the position IV the pull of the spring is equal to the weight and the acceleration is zero. The mass has its highest velocity at the point where it would come to rest under a gradually applied load. Beyond this point, represented by IV, the upward pull of the spring is greater than the weight and the body is negatively accelerated. It finally stops at the position of Fig. 182, V, with an elongation of the spring y_2. To calculate this elongation, we have:

$$Wy_2 = \frac{Ky_2^2}{2}, \tag{1}$$

$$y_2 = \frac{2W}{K} = 2y, \tag{2}$$

$$Ky_2 = 2W. \tag{3}$$

The deflection due to a suddenly applied load is twice as great as when the load is gradually applied, and the maximum force is twice the load. After reaching the maximum elongation the body vibrates back to its original starting point (provided the spring is perfectly elastic).

Fig. 182, VI, shows the mass W lifted a distance h above the position of II, in which it exerts no pull on the spring. If released suddenly, it falls this distance before it begins to stretch the spring. The total work done by gravity is the weight multiplied by the total distance $h + y$. At the lowest position VII this work has been transformed to energy of the spring.

$$W(h + y) = \frac{Ky^2}{2}. \tag{4}$$

$\therefore W(h+y) = \frac{Ky^2}{2} = \frac{P_{max} \cdot y}{2}$

Problems

1. A force of 8 pounds stretches a given spring 1 foot. A 6-pound mass is placed on the spring and gradually lowered. How much will the spring be stretched? *Ans.* 9 inches.

2. In Problem 1 the load is applied suddenly. What is the elongation of the spring and the maximum pull. *Ans.* 18 inches; 12 pounds.

3. In Problem 1 the load is lifted 1.44 feet and suddenly released. Find the maximum elongation and the maximum pull.

Ans. 2.4 feet; 19.2 pounds.

4. A spring board is made of plank 12 inches wide and 2 inches thick, and acts as a cantilever 10 feet long. Find the maximum unit stress when a boy weighing 60 pounds moves very slowly from the support to the free end.

Ans. 900 pounds per square inch.

5. Solve Problem 4 if the boy steps suddenly on the free end, neglecting the effect of the mass of the springboard upon its vibration.

Ans. 1,800 pounds.

6. Solve problem 5 if the boy jumps down on the end of the spring board from a height of 6 inches if the modulus of elasticity is 1,200,000 pounds per square inch and the effect of the mass of the spring is neglected.

It is evident that to determine the stress produced by a given load it is necessary to know how that load is applied.

Loads which are fixed in position and constant in magnitude are *dead loads.* The weight of a structure is a dead load. A load which is applied gradually, as the weight of falling snow, is treated as a dead load. A load which varies in position, such as the weight of a moving train on a bridge, is a *live load.* Any load which changes in position or magnitude will produce *impact stresses.* The magnitude of this impact factor depends upon the speed of application.

In most cases a varying load requires some time for its application, so that the stress produced by a live load is something less than twice that of an equal static load. The mass of the body subjected to load is also a factor which must be taken into account, as well as the natural time of vibration of the parts of which it is made up.

When a locomotive runs on a bridge, the effective unit stress produced may be 50 per cent. greater than that due to its weight alone. We say then that an impact factor of 50 per cent. should be added to the live load stress. If the speed is reduced, the impact factor is smaller.

166. Maxwell's Theorem.—In Chapter VIII it was taken for granted that the deflection at any point due to several loads at various points is the sum of the deflections due to each load taken separately. This is called Maxwell's theorem and may be regarded as an axiom which has been amply proven by experiment. Its use with the methods of external work leads to some interesting conclusions.

If A and B are two points on a beam the deflection at B due to a given load at A is equal to the deflection at A due to the same load at B.

Let y_a, Fig. 183, be the deflection at the point A when the load

P is placed at that point, and let y_{ab} be the deflection at B due to the load at A. Let y_b be the deflection at B when the load is at B, and y_{ba} be the deflection at A due to the load P at B.

Let the load P_1 be first placed on the beam at A and then let the equal load P_2 be placed at B. The work done at A when the load P_1 is applied is $\frac{P_1 y_a}{2}$. The point B is lowered a distance y_{ab} when the load P_1 is applied but no work is done since there is no force at B. Let the load P_2 be now applied at B producing a

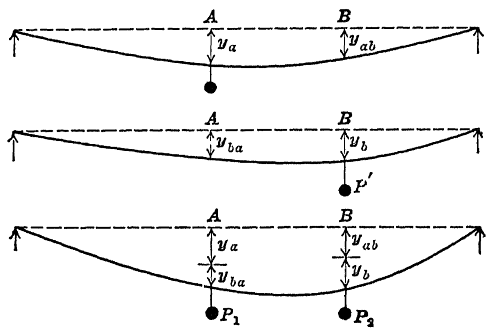


FIG. 183.

deflection y_b at B and a deflection y_{ba} at A. The work at B is $\frac{P_2 y_b}{2}$. At the same time the point A is deflected an additional y_{ba} under the full load of P_1, and the additional work is $P_1 y_{ba}$. The total work is

$$\frac{P_1 y_a}{2} + P_1 y_{ba} + \frac{P_2 y_b}{2}. \tag{1}$$

If the load P_2 be placed on B first and then the load P_1 on A the

$$\text{Work} = \frac{P_2 y_b}{2} + P_2 y_{ab} + \frac{P_1 y_a}{2}. \tag{2}$$

But equations (1) and (2) are equal and when $P_1 = P_2$ the first and last terms of (1) are identical with the last and first terms of (2) hence

$$P y_{ba} = P y_{ab}; \quad y_{ba} = y_{ab}. \tag{3}$$

Examples

Find the deflection at the middle of a beam supported at the ends due to a load at one-third the length from one end.

From equation (9) of Article 82,

$$EIy = \frac{Px^3}{12} - \frac{Pl^2x}{16}.$$

When

$$x = \frac{l}{3};\ EIy = \frac{23\,Pl^3}{1{,}296},$$

which gives the deflection at $\frac{l}{3}$ due to the load P at the middle. The deflection at the middle due to the load P at the third point is the same.

Find the deflection at the end of a cantilever due to a load uniformly distributed over a length a measured from the free end with no load on the remainder.

The load on a length dx is wdx. The deflection at a distance x from the free end due to a load wdx on the free end is,

$$EIy = \frac{wdx}{6}\,(x^3 - 3\,l^2x + 2\,l^3), \tag{4}$$

The deflection at the end due to the load wdx at x is the same, and the total deflection at the end due to the load on the length a is given by,

$$EIy = -\frac{w}{6}\int_0^a (x^3 - 3\,l^2x + 2\,l^3)\,dx = -\frac{w}{6}\left[\frac{x^4}{4} - \frac{3\,l^2x^2}{2} + 2\,l^3x\right]_0^a.$$

$$EIy = -\frac{w}{6}\left(\frac{a^4}{4} - \frac{3\,l^2a^2}{2} + 2\,l^3a\right).$$

When $a = l$, equation (6) becomes $EIy = -\frac{wl^4}{8}$.

For experiments on the impact of moving trains, see F. E. Turneaure, *Transactions American Society of Civil Engineers*, vol. XLI, pages 410–466.

CHAPTER XVI

COMBINED STRESS

167. Resultant of Shearing and Tensile Stress.—Fig. 184 represents a block of breadth dx, height dy, and length l, subjected to tensile stresses of intensity s_t perpendicular to the left and right vertical faces, to shearing stresses of intensity s_s parallel to these faces, and to shearing stresses of equal intensity in the top and bottom faces. The shear on the left face is upward and on the top face toward the left. It is desired to find the unit shearing stress parallel to the diagonal BG or CF and the unit

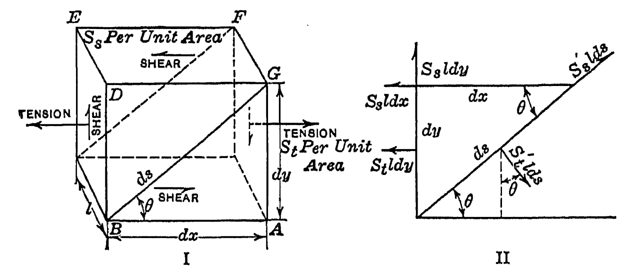


FIG. 184.—Combined shear and tension.

tensile stress normal to the plane $BCFG$. The block may be considered as divided by the plane $BCFG$ into two equal triangular prisms. The prism which lies to the left of this plane will be taken as the free body in equilibrium. The forces which act on this free body are five in number:

Total tension $s_t ldy$, toward the left, applied at center of $BCED$;

Total shear $s_s ldy$, upward, applied at center of $BCED$;

Total shear $s_s ldx$, toward the left, applied at center of $DEFG$;

Total shear on $BCFG$, parallel to BG, applied at center of $BCFG$;

Total tension normal to $BCFG$ at its center.

The unknown unit shearing stress in the plane $BCFG$ will be represented by s_s' and the unknown unit tensile stress by s_t'. The total shear on this plane is then $s_s' lds$, where ds is the length of the diagonal BG. The total tension on the diagonal plane is

$\leftarrow$ 100 # S_s

S_t 4000

$s_t'lds$. We will determine the magnitude of these unknown forces by resolving parallel to BG and normal to the plane $BCFG$. These five forces are represented in a single plane in Fig. 184, II. Resolving parallel to BG and dividing by l,

$$s_t dy \cos\theta + s_s dx \cos\theta - s_s dy \sin\theta = s_s' ds, \tag{1}$$

where θ is the angle between the plane $BCFG$ and the horizontal. Dividing by ds and substituting for $\frac{dx}{ds}$ and $\frac{dy}{ds}$.

$$s_s' = s_t \sin\theta \cos\theta + s_s [\cos^2\theta - \sin^2\theta], \tag{2}$$

$$s_s' = s_t \frac{\sin 2\theta}{2} + s_s \cos 2\theta. \tag{3}$$

Resolving normal to ds:

$$s_t dy \sin\theta + s_s dx \sin\theta + s_s dy \cos\theta = s_t' ds, \tag{4}$$

$$s_t' = s_t \sin^2\theta + 2 s_s \sin\theta \cos\theta, \tag{5}$$

$$s_t' = s_t \frac{1 - \cos 2\theta}{2} + s_s \sin 2\theta. \tag{6}$$

These equations apply when the external shearing stresses in the block have the directions of Fig. 184. If the shear is reversed some of the signs are changed.

Problems

1. With the unit shearing stress 100 pounds per square inch and the unit tensile stress in the same direction 400 pounds per square inch (Fig. 185), find the resultant unit shearing stress along a plane making an angle of 20 degrees with the direction of the tension. Also find the unit tensile stress normal to this plane. *Ans.* s_s', 205; s_t', 111 pounds per square inch.

2. With unit shearing stress 100 pounds per square inch and unit tensile stress zero, find the resultant tensile stress and shearing stress at 45 degrees. *Ans.* s_t', 100; s_s', 0.

168. Maximum Resultant Shearing Stress.—To find the direction that the plane $BCFG$ should have in order that the shearing stress along it shall be a maximum, differentiate the expression for s_s', Article 167, (3), with respect to θ:

$$\frac{d}{d\theta}(s_s') = s_t \cos 2\theta - 2 s_s \sin 2\theta = 0 \text{ for maximum or minimum} \tag{1}$$

$$\tan 2\theta_s = \frac{s_t}{2 s_s} = \frac{\frac{s_t}{2}}{s_s}. \tag{2}$$

For max. resulting s_s':

$\tan 2\theta = \frac{s_t}{2 s_s}$

The value of the maximum resultant unit shearing stress may be calculated by substituting the values of cos 2 θ and sin 2 θ in equation (3) of the preceding article. A right triangle may be formed with s_s as the base and $\frac{s_t}{2}$ as the altitude, Fig. 185. The angle adjacent to the side s_s is 2 θ, the hypotenuse is

$$\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}.$$

$$\operatorname{Cos} 2\,\theta = \frac{s_s}{\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}}, \quad \sin 2\,\theta = \frac{\frac{s_t}{2}}{\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}}. \qquad (3)$$

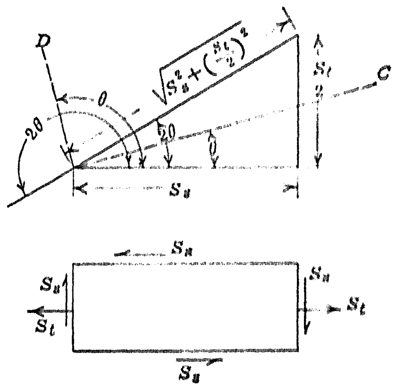


FIG. 185.—Maximum resultant shearing stress.

Substituting and dividing by the common factor,

$$\max s_s' = \pm\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}. \quad \text{Formula XXXI.}$$

The maximum unit shearing stress is the hypotenuse of a right triangle of which the unit shearing stress is one leg and one-half the unit tensile stress is the other. The direction of this maximum shearing stress makes an angle with the original tension and one of the original shears, which is one-half the angle of this triangle adjacent to the unit tensile stress.

For any given tangent there are two angles which differ by 180 degrees, consequently there are two values of 2 θ which are 180 degrees apart and two corresponding values of θ which are 90 degrees apart. These correspond to the two values of maximum shear at right angles to each other. These are in the direction of the lines OC and OD of Fig. 185.

Problems

1. A part of a solid is subjected to a horizontal tensile stress of 600 pounds per square inch and a horizontal and a vertical shearing stress of 400 pounds per square inch. Find the direction and magnitude of the maximum unit shearing stress.

Ans. $2\theta = 36° 52'$ or $216° 52'$; maximum unit shearing stress = 500 pounds per square inch at 18° 26′ and at 108° 26′ with the horizontal.

2. Find the maximum resultant shearing stress caused by a horizontal tensile stress of 800 pounds per square inch and a horizontal and vertical shearing stress of 400 pounds per square inch.

3. Solve Problem 1 for the magnitude of the maximum stress by means of equation (3) of Article 167.

4. In Problem 1 find the unit shearing stress at angles of 10 degrees, 20 degrees, 30 degrees, and 40 degrees with the horizontal, using equation (3) of the preceding article.

169. The Maximum Resultant Tensile Stress.—From equation (6) of Article 167,

$$s_t' = \frac{s_t}{2}(1 - \cos 2\theta) + s_s \sin 2\theta. \tag{1}$$

$$\frac{d}{d\theta}(s_t') = s_t \sin 2\theta + 2 s_s \cos 2\theta. \tag{2}$$

For the maximum and minimum s_t',

$$\tan 2\theta = -\frac{2 s_s}{s_t} = -\frac{s_s}{\frac{s_t}{2}} \tag{3}$$

Comparing with equation (2) of the preceding article it is seen that the double angle for maximum and minimum tensile stress is normal to the corresponding direction for maximum shear, and consequently the direction of maximum and minimum tension makes angles of 45 degrees with the directions of maximum shear.

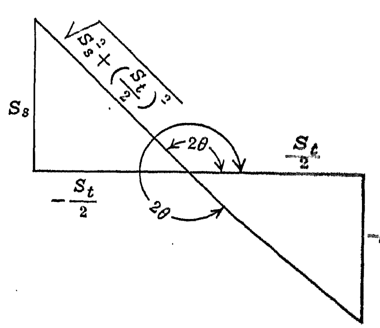


FIG. 186.—Double angle for maximum resultant tensile stress.

Using the double angle in the second quadrant (Fig. 186),

$$\sin 2\theta = \frac{s_s}{\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}}, \quad \cos 2\theta = -\frac{\frac{s_t}{2}}{\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}},$$

For Max. Resulta... S_t'

$\tan 2\theta = -\frac{2S_s}{S_t}$

$S_t' = \frac{S_t}{2} \pm S_s'$

which substituted in equation (1) gives,

$$\text{maximum } s_t' = \frac{s_t}{2} + \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2} = \frac{s_t}{2} + \max s_s'. \quad \text{Formula XXXII.}$$

Using the double angle in the fourth quadrant, the sine of 2 θ is negative and the cosine is positive. Substituting in equation (1),

$$\text{minimum } s_t' = \frac{s_t}{2} - \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2} = \frac{s_t}{2} - \max s_s'. \quad (4)$$

Since the maximum unit shearing stress is always equal to or greater than one-half the unit tensile stress, the second term of (4) is never less than the first term and the minimum stress is compressive.

Problems

1. In Problem 1 of the preceding article find the maximum and minimum unit tensile stress.

Ans. 800 pounds per square inch tensile stress; 200 pounds per square inch compressive stress.

2. Find the maximum resultant shearing and tensile stress due to a horizontal tension of 400 pounds per square inch and a horizontal and vertical shearing stress of 100 pounds per square inch.

Ans. $\tan 2\,\theta_s = 2$; $2\,\theta_s = 63° \, 26'$ or $243° \, 26'$.
$\theta_s = 31° \, 43'$ or $121° \, 43'$.
Max $s_s' = +223.60$ or -223.60 lb./in.^{2}
Max $s_t' = 423.60$ lb./in.^{2} tension.
Min $s_t' = 23.60$ lb./in.^{2} compression.

Fig. 187, II, shows the direction of the maximum resultant shearing stress for Problem 2. At 31 degrees 43 minutes the portion below the line exerts a shear to the right on the portion above. At 121 degrees 43 minutes the portion on the side of the line in which the angle is measured exerts a negative shear on the other side and the arrow representing positive shear (away from the origin) is on the other side. Fig. 187, III, shows how the shears act on the element of volume.

Fig. 187, IV, shows the direction of the maximum resultant tensile stress at a negative angle of 13 degrees 17 minutes, and a minimum stress of compression at 76 degrees 43 minutes.

Problem

3. Find the maximum resultant shearing and tensile stress due to a horizontal tension of 300 pounds per square inch and a horizontal and vertical shearing stress of 160 pounds per square inch.

Ans. Tan $2\,\theta_s = 0.9375$; $\theta_s = 21° \, 35'$ and $111° \, 35'$.
Max $s_s' = 219.32$ pounds.
Max $s_t = 369.32$; min $s_t' = -69.32$ pounds.

To find whether the maximum tensile stress is 45 degrees below or 45 degrees above the direction of the maximum shearing stress, consider Fig. 188, I. The tension due to the shear alone is 45 degrees below the horizontal at the right side. The tension resulting from this and the tensile stress of 300 pounds must lie between the two, and is, therefore, below the horizontal. To get the direction of the maximum unit tensile stress measure backward 45 degrees from the line at 21 degrees 35 minutes to the

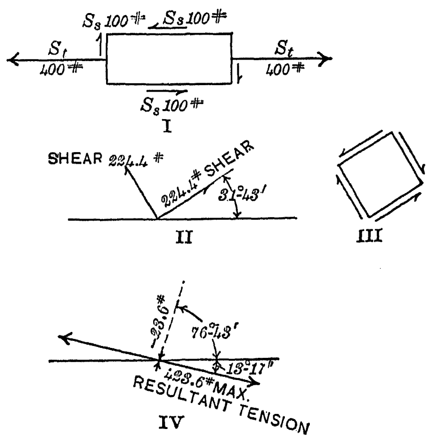


FIG. 187.—Shear and tension.

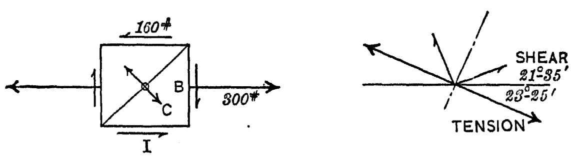


FIG. 188.—Direction of resultant tension.

negative direction of 23 degrees 25 minutes. The minimum tensile stress, which is a compression of 69.28 pounds per square inch, lies in the direction of the broken line in Fig. 188, II.

170. Resultant Stress in Beams.—In a beam the maximum resultant stress is due to a shearing stress which is a maximum at the neutral surface, and a tensile or compressive stress which is the greatest at the outer fibers. It is not usually necessary to calculate the maximum resultant tensile stress in a beam, since it is seldom greater than the bending stress in the outer fibers.

Problem

A 6-inch by 10-inch beam is supported at points 30 inches apart and carries a load of 20,000 pounds midway between the supports. Find the magnitude and direction of the maximum resultant tension, shear, and compression, at sections 5 inches and 10 inches from the left support at points 0, 1, 2, 3, 4, and 5 inches from the neutral axis.

Table XV, below, gives the results of the calculation for this problem. It will be noticed that the tension is at 45 degrees with the horizontal at the neutral surface and is 250 pounds per square inch. At 5 inches from the end the resultant tensile stress increases to 500 pounds per square inch in the outer fibers, and at 10 inches from the end it increases to 1,000 pounds per square inch.

TABLE XV. RESULTANT SHEAR AND TENSION IN A BEAM

Distance below axis		Shear	Tension	Maximum shear		Maximum tension		Maximum compression	
		Pounds	Pounds	Pounds	Angle	Pounds	Angle	Pounds	Angle
At 5 inches from end	0	250	0	250.0	0° 0′	250.0	−45° 0′	250.0	45° 0′
	1	240	100	245.2	5° 53′	295.2	−39° 07′	195.2	50° 53′
	2	210	200	232.6	12° 44′	332.6	−32° 16′	132.6	57° 44′
	3	160	300	219.3	21° 35′	369.3	−23° 25′	69.3	66° 35′
	4	90	400	219.0	32° 53′	419.0	−12° 07′	19.0	77° 53′
	5	0	500	250.0	45° 0′	500.0	0° 0′	0	90° 0′
At 10 inches from end	0	250	0	250.0	0° 0′	250.0	−45° 0′	250.0	45° 0′
	1	240	200	260.2	11° 49′	360.2	−33° 11′	160.2	56° 49′
	2	210	400	290.0	21° 48′	490.0	−23° 12′	90.0	66° 48′
	3	160	600	341.0	30° 58′	641.0	−14° 2′	41.0	75° 58′
	4	90	800	410.0	38° 40′	841.0	− 6° 20′	10.0	83° 40′
	5	0	1,000	500.0	45° 0′	1,000.0	0° 0′	0	90° 0′

The shearing stress is 250 pounds per square inch at the neutral axis at both sections, but due to the tensile stress it increases to 500 pounds per square inch in the outer fibers at the section 10 inches from the support.

Above the neutral axis the shear is the same as below and the tension and compression change places. The angles are numerically the same but are on opposite sides of the horizontal. Fig. 189 shows the direction and relative magnitude of the maximum and minimum stresses for this problem. Near the bottom where the maximum compression is small its direction is shown by the dotted lines. In the same way the direction of the tension is indicated near the top.

171. Bending Combined with Torsion.—In a shaft subjected to bending moment, the maximum tensile stress is found in the fibers at the dangerous section which are most remote from the neutral surface. When subjected to torsion, all the outer fibers are at the maximum shearing stress. When the shaft is subjected to the combined effect of bending moment and torque, those fibers at the dangerous section which are farthest from the neutral surface are subjected to the combined effect of the maxi-

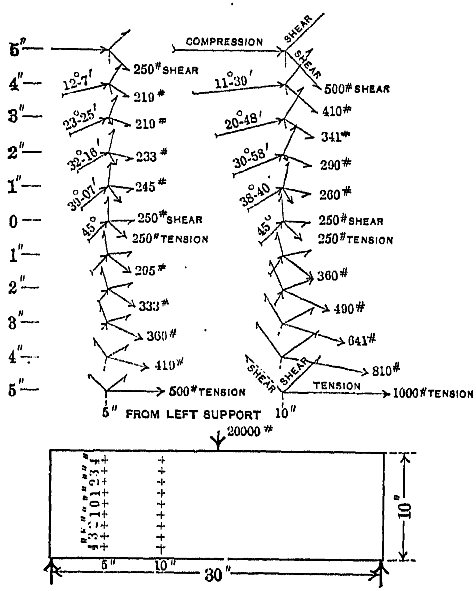


FIG. 189.—Resultant stress in a beam section.

mum tensile or compressive stress and the maximum shearing stress may be much larger than the results of Formulas VII and XIV.

Example

A 1-inch rod projects from a vise. A wrench, at right angles to the rod, grips it 8 inches from the vise. The wrench is turned by a force of 60 pounds, perpendicular to the plane of the rod and wrench, which is applied to the wrench 12 inches from the axis of the rod. Find the maximum resultant shearing and tensile stress.

The bending moment at the vise is the same as if the force of 60 pounds were applied directly to the rod at 8 inches from the vise, Fig. 190.

$$M = 60 \times 8 = 480 \text{ inch-pounds.}$$

$$S_t = \frac{480 \times 32}{\pi} = \frac{3{,}840 \times 4}{\pi} = 4{,}889 \text{ lb./in.}^2$$

$$T = 60 \times 12 = 720 \text{ inch-pounds.}$$

$$S_s = \frac{720 \times 16}{\pi} = \frac{3{,}840 \times 3}{\pi} = 3{,}667 \text{ lb./in.}^2$$

$$\text{Maximum } S_s' = \sqrt{3{,}667^2 + 2{,}444^2} = 4{,}407 \text{ lb./in.}^2$$

$$\text{Maximum } S_t' = 2{,}444 + 4{,}407 = 6{,}851 \text{ lb./in.}^2$$

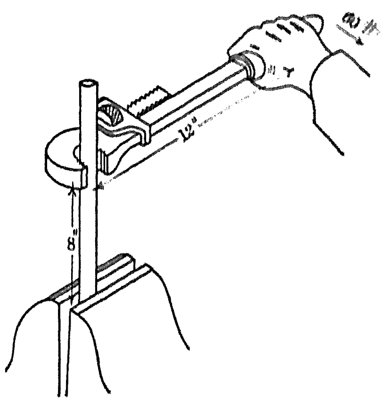


Fig. 190.—Torsion and bending.

Since the section modulus used in torsion is twice that used in bending, and the force P is the same for both torque and bending moment, there is a large common factor which may be taken out to reduce the labor of computation. In this problem the factor is $\frac{3{,}840}{\pi}$ which is equal to 1,222.

$$\text{Max } S_s' = 1{,}222\sqrt{3^2 + 2^2} = 1{,}222\sqrt{13} = 4{,}407 \text{ lb./in.}^2$$

Problems

1. A 2-inch round rod projects 2 feet from a vise and is twisted by a force of 400 pounds at the end of a 4-foot wrench. Find the maximum resultant shearing and tensile stress.

Ans. Max $S_s' = 13{,}665$ lb./in.^{2}; max $S_t' = 19{,}776$ lb./in.^{2}

2. A 4-inch solid shaft transmits 200 hp. at 120 r.p.m., and is subjected to a compression of 24,000 pounds parallel to its length. Find the maximum resultant compressive and shearing stress.

Ans. $S_s = 8{,}359$ lb./in.^{2}; max $S_s' = 8{,}413$ lb./in.^{2}
$S_c = 1{,}910$ lb./in.^{2}; max $S_c' = 9{,}368$ lb./in.^{2}

172. Equivalent Moment and Torque.—For a circular section $J = 2I$, and when the outer radius is a,

$$S_t = \frac{Ma}{I}, \quad \frac{S_t}{2} = \frac{Ma}{2I}; \tag{1}$$

$$S_s = \frac{Ta}{J} = \frac{Ta}{2I}, \tag{2}$$

$$\text{Max } S_s' = \sqrt{S_s^2 + \left(\frac{S_t}{2}\right)^2} = \frac{a}{2I}\sqrt{T^2 + M^2} = \frac{a\sqrt{T^2 + M^2}}{J}. \tag{3}$$

The term $\sqrt{T^2 + M^2}$ may be regarded as the equivalent torque resulting from the combination of torsion and bending. In the example of the preceding article $M = 480$ inch-pounds, $T = 720$ inch-pounds and the equivalent torque is $240\sqrt{13} = 865.3$.

$$\text{Max } S_s' = \frac{865.3 \times 16}{\pi} = 4{,}407 \text{ lb./in.}^2$$

$$\text{Max } S_t' = \frac{Ma}{2I} + \frac{a\sqrt{T_2 + M_2}}{2I} = a\frac{(M + \sqrt{T^2 + M^2})}{2I}. \tag{4}$$

The term $\dfrac{M + \sqrt{T^2 + M^2}}{2}$ may be regarded as the equivalent bending moment.

Problem

A hollow shaft of 4 inches inside diameter and 6 inches outside diameter is subjected to a torque of 2,000 foot-pounds and a bending moment of 1,500 foot-pounds. Find the equivalent maximum torque and moment and find the maximum unit shearing and tensile stress.

173. Shear Combined with Tension in Two Directions.—Fig. 191 represents a block subjected to shearing stress and horizontal tension as in the case of Fig. 184, with an additional vertical tension of intensity s_v. Dividing the block by means of a plane at angle θ with the horizontal, and taking either half as the free body in equilibrium, the intensity of the shearing stress in this inclined surface is found by resolving parallel to it. If the length of the block normal to the plane of the paper is unity,

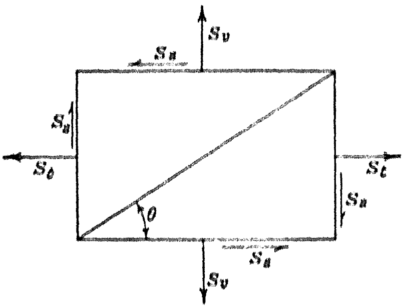


FIG. 191.—Tension in two directions combined with shear.

$$s_s' ds = s_t dy \cos\theta + s_s dx \cos\theta - s_s dy \sin\theta - s_v dx \sin\theta. \tag{1}$$

Dividing by ds and substituting $\frac{dx}{ds} = \cos\theta$, $\frac{dy}{ds} = \sin\theta$,

$$s_s = (s_t - s_v)\sin\theta\cos\theta + s_s(\cos^2\theta - \sin^2\theta); \tag{2}$$

$$s_s = \frac{s_t - s_v}{2}\sin 2\theta + s_s\cos 2\theta. \tag{3}$$

Equation (3) is the same as equation (3) of Article 167 with $s_t - s_v$ in place of s_t.

Differentiating (3) with respect to θ and equating to zero to get the direction of maximum unit shearing stress,

$$(s_t - s_v)\cos 2\theta - 2\,s_s\sin 2\theta = 0; \tag{4}$$

$$\tan 2\theta = \frac{\frac{s_t - s_v}{2}}{s_s}. \tag{5}$$

Solving for the sine and cosine of 2θ and substituting in (3),

$$\text{Max } s_s = \sqrt{s_s^2 + \left(\frac{s_t - s_v}{2}\right)^2}, \tag{6}$$

which is the same as Formula XXXI with $s_t - s_v$ in place of s_t. Resolving perpendicular to ds,

$$s\,ds = s_t dy\sin\theta + s_s dx\sin\theta + s_s dy\cos\theta + s_v dx\cos\theta; \tag{7}$$

$$s' = s_t\sin^2\theta + s_v\cos^2\theta + 2\,s_s\sin\theta\cos\theta; \tag{8}$$

$$s' = \frac{s_t}{2}(1 - \cos 2\theta) + \frac{s_v}{2}(1 + \cos 2\theta) + s_s\sin 2\theta. \tag{9}$$

For the direction of maximum unit tensile stress,

$$\tan 2\theta = -\frac{s_s}{\frac{s_t - s_v}{2}}, \tag{10}$$

which is normal to the double angle for the maximum shearing stress.

$$\text{Max } s_t' = \frac{s_t + s_v}{2} + \sqrt{s_s^2 + \left(\frac{s_t - s_v}{2}\right)^2}. \tag{11}$$

If the vertical stress is compressive it may be regarded as negative tension. If s_c is this compressive stress,

$$\text{Max } s_s' = \sqrt{s_s^2 + \left(\frac{s_t + s_c}{2}\right)^2}; \tag{12}$$

$$\text{Max } s_t' = \frac{s_t - s_c}{2} + \text{max } s_s'. \tag{13}$$

If s_s is zero, equation (6) gives $\frac{s_t - s_v}{2}$ as the maximum unit shearing stress at 45 degrees with both s_s and s_v. There is a greater unit shearing stress of magnitude $\frac{s_t}{2}$ in a plane parallel to s_v, at an angle of 45 degrees with the direction of the greater stress, s_t.

Equation (11) shows that when the shearing stress is zero, the maximum tensile stress is s_t.

Problems

1. A block is subjected to a horizontal tensile stress of 600 pounds per square inch, vertical compressive stress of 200 pounds per square inch, and horizontal and vertical shearing stress of 300 pounds per square inch. Find the maximum unit shearing and tensile stress.

Ans. Max $s_s' = 500$ lb./in.^{2}; max $s' = 700$ lb./in.^{2}

2. A 1-inch round rod projects from a vise and is twisted by a force of 60 pounds at the end of a 12-inch wrench. The pressure at the jaws of the vise is 4,000 pounds per square inch. Find the maximum stresses if the wrench is applied 10 inches from the vise and the direction of the force of 60 pounds is normal to the plane of the jaws.

Ans. Max $s_s' = 6{,}245$ lb./in.^{2}; max $t_t' = 7{,}300$ lb./in.^{2}

3. A block is subjected to a horizontal tensile stress of 600 pounds per square inch, and a vertical tensile stress of 200 pounds per square inch, together with horizontal and vertical shearing stress of 300 pounds per square inch in the plane of the two tensile stresses. Find the maximum unit shearing stress. *Ans.* Max $s_s' = 360.6$ pounds per square inch.

4. Solve Problem 3 if the unit shearing stress be only 100 pounds per square inch. *Ans.* Max $s_s' = 300$ pounds per square inch.

CHAPTER XVII

THEORIES OF ELASTIC LIMIT AND FAILURE

174. Principles Involved.—In the preceding chapter, methods are given for finding the maximum tensile, compressive, and shearing stresses developed by a combination of stresses. It is a question which of these stresses determines the elastic limit and the failure. Also in Chapter I it was shown that stress in one direction causes a deformation in the opposite sense in all directions at right angles to the direct applied force. For instance, if there is a compressive stress along the X axis producing unit deformation δ, there is unit elongation $\sigma\delta$ along the Y and Z axes. If, at the same time, there is a tensile stress along the Y axis, the total elongation along that axis is that due to the tension in its direction in addition to the elongation due to the compression along the X axis. It is a question whether the failure which may occur at right angles to the tensile stress is influenced in any way by the additional elongation due to the compression.

As a result of these various considerations there are several theories to account for the relation of the stresses to the elastic limit and the failure.

175. The Maximum Stress Theory.—The *maximum stress theory*, called also *Rankine's* theory, assumes that failure is due to the single stress which is the largest, without reference to other stresses at angles thereto, except insofar as the components of these stresses affect the value of the maximum unit stress. If a block is subjected to a tensile stress s_t and to a tensile stress s_v at right angles thereto, if s_t is greater than s_v the maximum stress is equal to s_t. This may be shown by resolution as in Article 173 or by means of equation (11) of that article by setting s_s equal to zero. According to the maximum-stress theory the block will fail by rupture along a plane approximately normal to the maximum stress when this stress s_t reaches the ultimate strength of the material and the value of s_t to produce rupture is independent of the other stress s_v.

While this theory can hardly be said to be accepted by any one who seriously considers the subject, it is, nevertheless considerably used in practice. In a boiler, for instance, the circumfer-

ential tensile stress tending to rupture the shell longitudinally is twice as great as the longitudinal tensile stress, and it is customary to calculate from the circumferential stress alone without reference to the other.

176. The Maximum Strain Theory.—This theory, which is also called *Saint Venant's* theory, assumes that a solid reaches its elastic limit when the unit deformation reaches a given limit and that there is an ultimate unit deformation which cannot be exceeded without rupture no matter in what way the stresses are applied which cause the deformation.

Suppose a block, Fig. 192, is subjected to a direct tensile stress of s_t and to a compressive stress at right angles thereto of s_c, and suppose the material reaches its elastic limit in tension when the unit elongation is 0.001. According to the maximum strain theory, if the unit elongation due to the tension is 0.0008 and there is unit elongation in the same direction of 0.0002 due to the transverse compression, the elastic limit will be reached. Also, if the material will stand only a small unit elongation compared with the unit compression, it will fail by splitting along surfaces which are parallel to the direction of the compressive load.

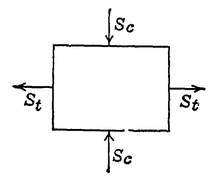


FIG. 192.—Tension and compression of right angles.

177. The Maximum Shear Theory.—This is frequently called the Guest* theory or the Guest-Hancock† theory. According to this theory, a given material reaches its elastic limit in tension or compression, when the unit shearing stress, as calculated by equation (12) of Article 173 or by Formula XXXI, reaches the elastic limit of the material in shear, and failure occurs when the unit shearing stress, as calculated by these formulas, reaches the ultimate shearing strength of the material.

As stated above there is no question as to the truth of the theory. Fig. 193 shows three wooden blocks which were tested in compression. Failure has taken place by shear along planes at about 45 degrees with the direction of the stress, at which angle

* See J. J. GUEST, "On The Strength of Ductile Materials Under Combined Stress," *Philosophical Magazine*, July, 1900, pages 69–132.

† E. L. HANCOCK, "The Effect of Combined Stress on the Elastic Properties of Steel," *Proceedings of the American Society for Testing Materials*, 1905, pages 179–186; 1906, pages 295–307.

the unit shearing stress is a maximum. In a tensile test of soft steel the edges at the fracture are inclined at an angle of approximately 45 degrees to form the so-called crater.

It is evident that a bar in tension or compression will fail by shear provided it does not fail in some other way before the unit

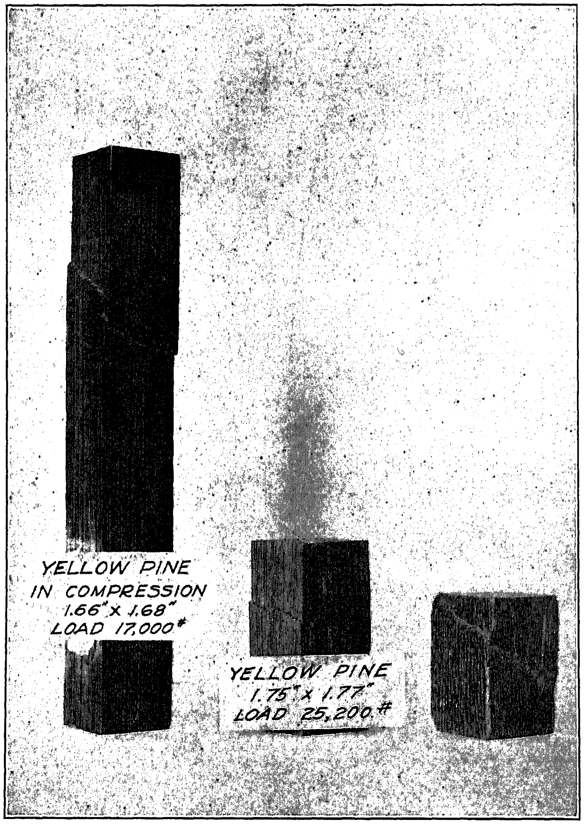


FIG. 193.—Timber in compression.

shearing stress (which, at 45 degrees, is one-half the unit tensile or compressive stress) reaches the ultimate shearing strength of the material. It is also evident that when the unit shearing stress reaches the elastic limit in shear, there will be large linear deformations which will appear as the elastic limit in tension or compression. The point upon which there is *not agreement* is whether a solid ever reaches its elastic limit in tension or com-

pression before reaching the elastic limit in shear, and whether failure is always by shear.

The tests made by Guest and Hancock were upon ductile materials, and neither of them claimed that the maximum shear theory applies to brittle solids.

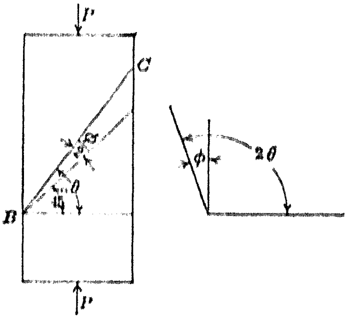


FIG. 194.—Shear failure caused by compression.

To find the angle of failure by shear due to compression when there is considerable friction, consider Fig. 194. The component of the load P along the plane BC at an angle θ with the normal to the applied force, is $P \sin \theta$; the component normal to this plane is $P \cos \theta$. If A is the area of the normal cross-section the area of the plane BC is $A \sec \theta$; and if s_s is the unit shearing strength, the shearing resistance is $s_sA \sec \theta$. Resolving parallel to BC,

$$P \sin \theta = P\mu \cos \theta + s_sA \sec \theta, \tag{1}$$

where μ is the coefficient of friction.

$$\frac{P}{s_sA} = \frac{\sec \theta}{\sin \theta - \mu \cos \theta}, \tag{2}$$

$$\frac{s_sA}{P} = \sin \theta \cos \theta - \mu \cos^2 \theta,$$

$$\frac{2\, s_sA}{P} = \sin 2\,\theta - \mu\,(1 + \cos 2\,\theta). \tag{3}$$

The load P is a minimum when the second member of (3) is a maximum. Differentiating with respect to θ and equating to zero,

$$\cos 2\,\theta + \mu \sin 2\,\theta = 0;$$

$$\cot 2\,\theta = -\mu;\ 2\,\theta = 90° + \text{arc} \tan \mu;$$

$$\theta = 45° + \frac{\text{arc} \tan \mu}{2} \tag{4}$$

Failure takes place along a plane which makes an angle of 45 degrees plus one-half the angle of friction with the plane normal to the compressive force.

178. Failure.—As previously stated, failure of ductile material in tension generally takes place by shearing at about 45 degrees

to the direction of the tensile stress. Non-ductile material such as cast iron or porcelain, fails at right angles to the direction of the tensile stress. The shearing stress at 45 degrees due to tension is one-half of the unit tensile stress, so that failure by shear indicates that the shearing strength is less than one-half of the tensile stress. Fig. 195 shows a wooden rod which was tested in tension; the fracture is zigzag, indicating shear at angles considerably greater than 45 degrees. Timber has small shearing strength parallel to the grain, which accounts for this kind of failure. In compression, timber fails by shear at about 45 degrees as shown in Fig. 193. The blocks shown in Fig. 193 are exceptional in that the shear occurs for a considerable distance in a single plane. Generally there is shear for a short distance then splitting to another shear plane.

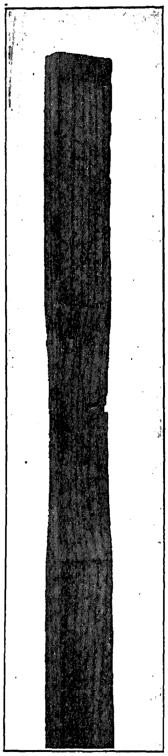


FIG. 195.—Timber in tension.

Fig. 196 shows pieces of wrought-iron pipe which have been tested in compression. The material flows under stress causing considerable enlargement, and finally splits.

Fig. 197 is hard brick in compression. The failure takes place at an angle much greater than 45 degrees. The increased angle is due to the friction of the material. In the case of timber the angle is practically 45 degrees, probably because the shear takes place by bending of the fibers at the shear plane instead of by sliding.

Fig. 198 shows two 4-inch by 4-inch blocks of 1:1 cement mortar, each of which failed by shearing at the ends and then splitting lengthwise. The longitudinal fracture may be explained as due to the wedge action of the shear pyramids. Another explanation is that this failure is due to the lateral expansion caused by the longitudinal pressure. If Poisson's ratio is 0.15 and the compressive stress is 4,000 pounds per square inch (which was about the stress in these prisms), there is a lateral expansion which is equivalent to the elongation caused by a tensile stress of 600 pounds per square inch, which easily accounts for the failure by the maximum-strain theory. An explanation for the

pyramids and shear action at the ends is that the friction of the heads of the testing machine prevented splitting here.

Porcelain rods 16 inches long and 1 inch in diameter failed by splitting nearly the entire length with no sign of shear at

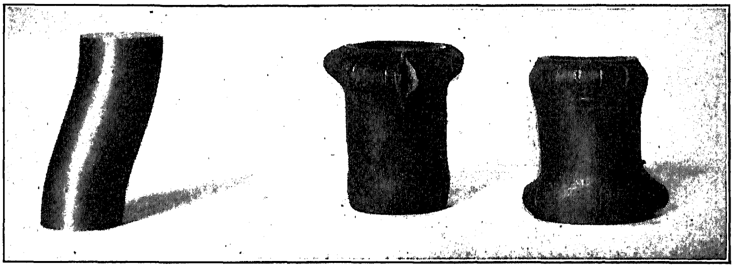


FIG. 196.—Metal in compression.

the ends, indicating that, for brittle material with relatively small tensile strength, the maximum strain theory holds.

The bearing strength of a solid depends upon the relative size of the surface of contact and the entire dimensions of the body.

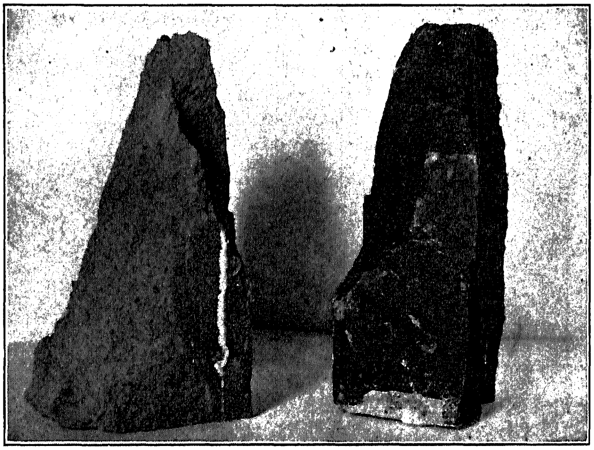


FIG. 197.—Hard brick in compression.

In the treatment of bearing stress there are two limiting cases. The first is that shown in Fig. 199 in which the surface of contact is equal to the entire cross-section of the body B, and the length in the direction of the applied force is at least equal to the thickness of the body. In this case the bearing strength is equivalent

to the compressive strength. Used in this way, a soft material like babbitt metal would show little bearing strength. Fig. 200 shows a second case. Here the load is applied to a small portion of the body which is of unlimited extent or is confined laterally by another body. The portion outside of the loaded area acts as a hoop to prevent the lateral expansion. In this form, a body

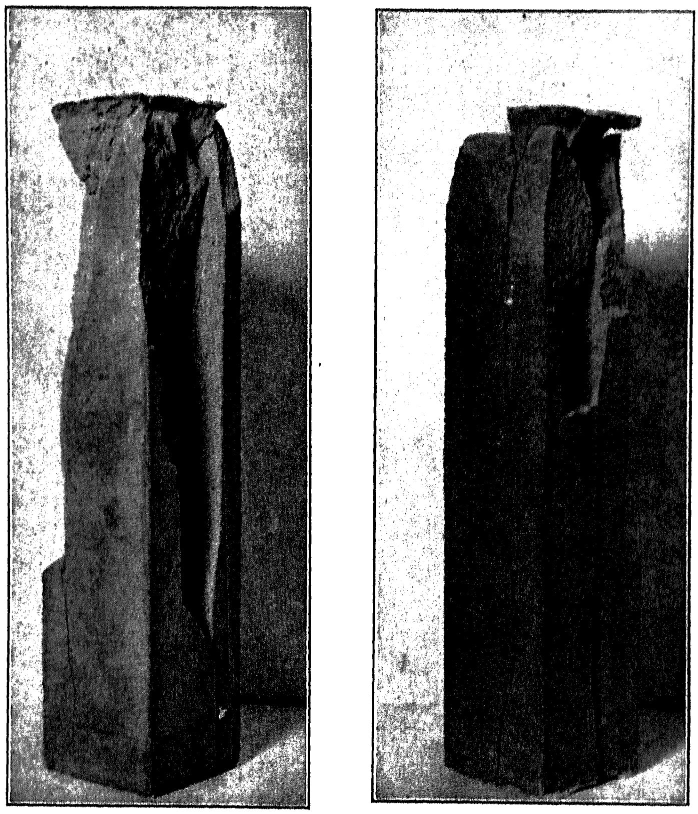


FIG. 198.—Cement in compression.

composed of *separate particles* may have considerable bearing strength, depending upon the friction. Dry sand is an example. In a mass of wheat or flaxseed, where the friction is smaller, the bearing strength is less.

Fig. 201 shows two cases intermediate between Figs. 199 and 200.

Cutting with a knife or chisel depends upon the bearing

strength of the tool and of the material cut. The bearing strength of the tool under the conditions of Fig. 199 must be greater than that of the material under the conditions of Fig.

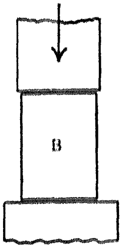


FIG. 199.—Bearing.

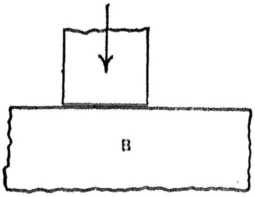


FIG. 200.—Bearing on large body.

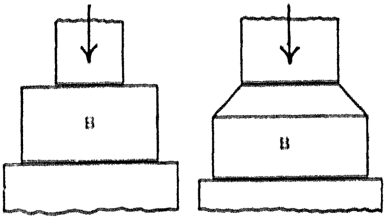
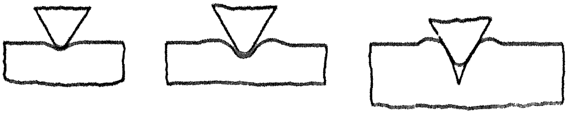


FIG. 201.—Cases of bearing pressure.

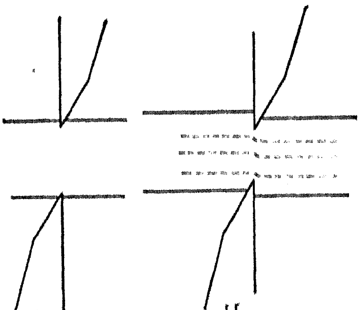
200. At first there is a depression in the material under the edge of the tool, as shown in Fig. 202, I. When the unit stress in



I II III

FIG. 202.—Cutting.

the material exceeds the bearing strength, it is permanently pushed back. In a plastic non-porous material, some of the substance is forced up by the pressure, as shown in Fig. 202, II.



I II

FIG. 203.—Cutting with shears.

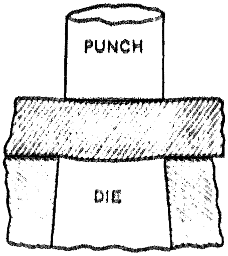


FIG. 204.—Punching a plate.

In a porous body like wood there is an increase in density adjacent to the cutting surface. The wheel of a wagon cutting in soft earth illustrates both cases. If the earth is wet clay, we

have an illustration of the plastic non-porous substance; if it is dry loam, it approaches the other case.

When a cutting tool has penetrated a little distance, it acts as a wedge and exerts a tensile stress upon the material in front of its edge. This is shown in Fig. 202, III.

Fig. 203 shows the behavior of a pair of scissors or shears. At the beginning, the cutting is due to the bearing stress on the cutting edges, as shown in Fig. 203, I. As the edges penetrate

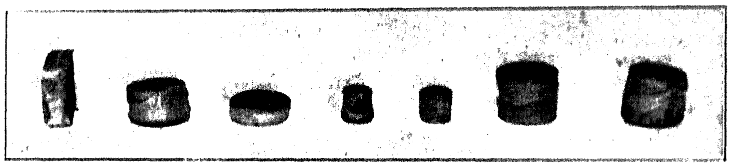


FIG. 205.—Slugs punched from steel plates.

into the material the bearing force is increased at each blade. These forces produce shearing stresses in all portions of the body in the plane of the cutting edges. The corresponding shearing deformations are shown by the dotted lines in Fig. 203, II. Fig. 204 represents the punching of a metal plate. The plate is bent a little at first, which makes the surface of contact a narrow ring at the edge of the punch and die. When the compressive stress on these rings exceeds the bearing strength of the plate, cutting begins. This is followed by shear, as in the case of cutting with scissors.

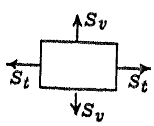


FIG. 206.—Tension in two directions.

Fig. 205 shows some of the slugs punched from steel plates. Notice the curvature at the ends. In the case of the small diameter compared with the length, the punch failed after making about a dozen holes.

179. Biaxial Loading.—The most common form of biaxial loading consists of two tensions, one tension and one compression, or two compressions at right angles to each other. Fig. 206 represents two tensile stresses of intensities s_t and s_v. In this case the maximum unit shearing stress is $\frac{s_t}{2}$ no matter how great s_v may be, provided it is not greater than s_t. If the maximum shear theory holds, the material subjected to biaxial loading should reach its elastic limit at the same values of s_t no matter what is the value of s_v, or whether s_v be tension or compression. On the

other hand, a tensile stress s_v will diminish the unit strain in the direction of s_t and a compressive stress in the direction of s_v will increase the unit strain in the direction of s_t. The unit strain in the direction of s_t is given by

$$\delta = \frac{s_t}{E} - \frac{s_v \sigma}{E}. \tag{1}$$

If the maximum-strain theory be the correct one, the unit stress in the direction of s_t at the elastic limit or yield point will be increased as s_v is increased, and will be diminished if s_v is changed to compression.

Prof. Albert J. Becker* has performed an extensive series of experiments with biaxial loading to determine these points.

These tests were made on hollow steel cylinders. The pressure of a liquid inside these cylinders produced a circumferential tensile stress, and an axial tensile stress. The axial stress was further increased by the direct pull of a testing machine. In each test the ratio of the circumferential unit stress to the axial unit stress was kept constant. Johnson's apparent elastic limit for the axial deformation was taken as the limit to be determined.

Table XVI gives approximately the results of one set of experiments.†

TABLE XVI. BIAXIAL LOADING TESTS

Tube number	Ratio of circumferential stress to axial stress	At apparent elastic limit	
		Axial unit stress, lb./in.^{2}	Axial unit elongation
5	0.0	43,000	0.00165
1	0.240	46,000	0.00168
2	0.475	50,000	0.00152
4	0.69	50,000	0.00152
3	0.92	50,000	0.00140

Tube No. 5, with no circumferential stress, reached the elastic limit at the tensile stress of 43,000 pounds per square inch. The maximum unit shearing stress at 45 degrees was 21,500 pounds per square inch. If failure always takes place by shear, the other

* A. J. BECKER, "The Strength and Stiffness of Steel under Biaxial Loading," *Bulletin* No. 85 of The University of Illinois Engineering Experiment Station.

† The data of Table XVI were estimated from the curves of Fig. 17 of *Bulletin* No. 85 of the University of Illinois Engineering Experiment Station.

tubes should reach the elastic limit at the same unit stress. It is seen, however, that tube No. 1 reached the elastic limit at about 46,000 per square inch axial stress, and the others at about 50,000 per square inch. It will be noted that the axial unit deformation at the elastic limit is about the same for the first two tubes and is less for the other three. It is evident that there are two sets of limiting conditions which determine the elastic limit. The material reaches its elastic limit when the unit deformation is about 0.00166; it also reaches the elastic limit when the unit shearing stress becomes about 25,000 pounds per square inch. Tubes 5 and 1 reached the limiting deformation before reaching the limiting shearing stress. In the other tubes, the greater transverse tension reduced the axial deformation so that they reached the limiting unit shearing stress while the unit strain was still considerably below the limit.

From the entire series of tests, of which Table XVI is only a small part, Becker concludes:*

> "For increasing values of the ratio of the biaxial stresses the yield-point strength follows the maximum-strain theory until the value of the shearing stress reaches the shearing yield point, then the shearing stress controls according to the maximum-shear theory. *There are thus two independent laws each dominant within proper limits instead of some single law as has heretofore been assumed.*"

Table XVI shows that the elastic limit in shear is about 60 per cent. of the elastic limit in tension. If the maximum-shear theory were true in all cases it would mean that the elastic limit in shear is always 50 per cent. of the elastic limit in tension.

In thin cylinders, such as boilers, the longitudinal unit stress is one-half of the circumferential unit stress, and both are tension. If s_t is the unit circumferential stress, the unit longitudinal stress is $\frac{s_t}{2}$. The circumferential unit deformation is reduced by the longitudinal unit stress. If Poisson's ratio is $\frac{1}{4}$ the unit deformation circumferentially is

$$\delta = \frac{s_t - \frac{s_t}{8}}{E} = \frac{7}{8}\frac{s_t}{E}, \tag{2}$$

so that the actual unit deformation is only seven-eighths as much

**Bulletin* No. 85, Illinois University Engineering Experiment Station, page 85.

as that which would be produced by the circumferential unit stress acting alone. It is not customary to consider this in calculating the strength of boilers. The error, when it is neglected, is on the side of safety. The error is really small, for the weak part of a boiler is at a joint. At a longitudinal joint, on account of the lapping of the plates or the butt straps, there is much more material to resist longitudinal tension than in the main plates, and the longitudinal unit stress is less than $\frac{s_t}{2}$. Also, there is an unequal distribution of both stresses on account of the material cut away for the rivet holes which produces an error on the side of danger. For these reasons it is not advisable to make any allowance for the reduced strain due to the combined stress.

180. Combined Tension and Shear.—It was shown in Article 169 that tensile stress combined with shearing stress parallel and perpendicular thereto gives:

$$\text{Max } s_t' = \frac{s_t}{2} + \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}; \quad \text{Formula XXXII.}$$

$$\text{Min } s_t' = \frac{s_t}{2} - \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}; \tag{2}$$

The minimum is equivalent to a compressive stress

$$s_c' = -\frac{s_t}{2} + \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}. \tag{3}$$

The unit deformation in the direction of the tensile stress is that produced by the maximum tensile stress plus the effect of the compressive stress at right angles thereto.

$$E\delta = \frac{s_t}{2} + \sqrt{s^2 + \left(\frac{s_t}{2}\right)^2} + \sigma\left(-\frac{s_t}{2} + \sqrt{s^2 + \left(\frac{s_t}{2}\right)^2}\right); \tag{4}$$

$$E\delta = \frac{s_t}{2}(1 - \sigma) + (1 + \sigma)\sqrt{s^2 + \left(\frac{s_t}{2}\right)^2}. \tag{5}$$

When Poisson's ratio is $\frac{1}{4}$

$$E\delta = \frac{3}{8}\frac{s_t}{} + \frac{5}{4}\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}. \tag{6}$$

It is well known that brittle materials, under these conditions, fail by tension. A cast-iron rod broken by torsion, or by torsion and bending combined, fails along a curve which is approximately normal to the maximum tensile stress.

The same is true of concrete. Fig. 217 shows a characteristic failure of a reinforced-concrete beam supported at the ends and loaded at the third points. A diagonal crack starts near the bottom and runs up to the point of application of one load. Such cracks are usually found in reinforced-concrete beams which are

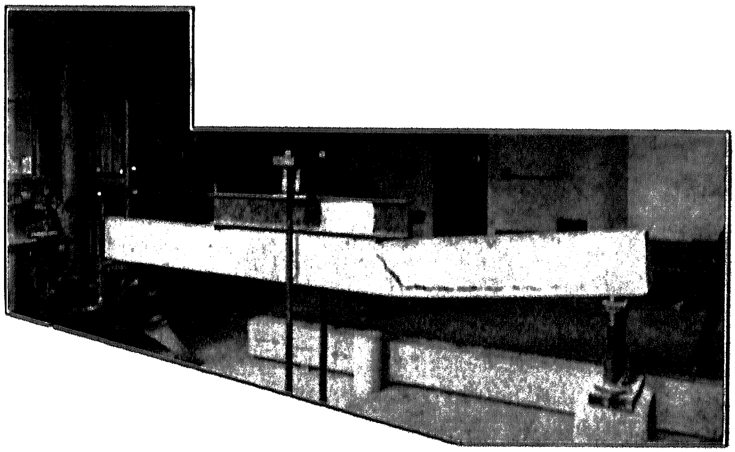


FIG. 217.—Failure of a reinforced-concrete beam.

tested in such a way as to develop large shear. Between the two concentrated loads the shear is zero (except that due to the weight of the beam) and the cracks in this part of the beam are vertical. The large crack in Fig. 207 also extends horizontally along the line of the reinforcement but this part only opened as the beam approached total failure, while the diagonal crack is one of a number which appeared at about one-half the ultimate load.

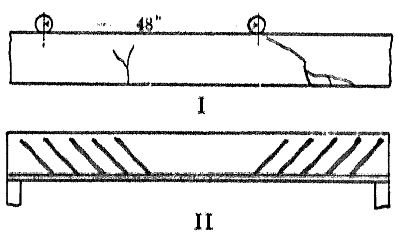


FIG. 208.

Fig. 208, I is a drawing showing the same effect. The crack between the loads, running nearly vertical is called a tension crack, while the one on the right is called a shear crack. Both are really tension cracks but the right one is due to the tension resulting from the combination of tension and shear.

To prevent failure by combined tension and shear, beams are reinforced as shown in Fig. 208, II. The diagonal bars are called

shear bars and are placed in approximately the direction of the maximum resultant tensile stress and normal to the direction of the so-called shear cracks.

It is not known whether these are the effects of maximum stress or maximum strain. There is not a great relative difference between the maximum stress (Formula XXXII) and the stress which corresponds with the maximum strain (equations (5) or (6)) so that careful measurements will be necessary to differentiate between them. In calculations of this kind to find the allowable load it is customary to use Formula XXXII.

Combined tension and shear have been used to test the theories of failure as applied to ductile materials. J. J. Guest* tested cylinders of soft steel, iron, brass, and copper by combined tension and torsion. Some of these cylinders were solid but most of them were thin hollow tubes. The internal pressure of a liquid was also applied to some of the tubes. He determined the *yield point in tension* in the cylinder thus subjected to combined stress and came to the conclusion that the yield point is reached when the resultant shearing stress as calculated by Formula XXXI reaches a definite value. These researches, which are regarded as classic, are open to the criticism that several sets of determinations were made on the same tube, so that the elevation of the yield point when the material is stressed to its yield point becomes a disturbing factor.

Later, E. L. Hancock tested hollow steel cylinders under combined shear and tension. His experiments show definitely that, "The presence of a torsional stress lowers the unit stress and the unit strain at the elastic limit in tension and also lowers the modulus of elasticity, somewhat."*

Hancock calculated what he called the true tensile stress, by equation (5) for Poisson's ratios of ¼ and ⅓. He also calculated a value, which he called the true shearing stress, by the formula $(1 + \sigma)\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}$, and arrived at the conclusion that this "true unit shearing stress" determines the elastic limit of the material.†

In the present state of our knowledge of combined shear and tension or compression it is best to calculate the unit shearing stress by Formula XXXI and the unit tensile stress by Formula

* *Philosophical Magazine*, July, 1900.

† *Proceedings of the American Society for Testing Materials*, 1906, page 301.

XXXII and see that neither of these exceeds the allowable unit stress of its kind. In place of the tensile stress by Formula XXXII, the stress which corresponds with the maximum unit strain may be computed by equation (5).

Probelms

1. The allowable unit tensile stress is 15,000 and the allowable unit shearing stress is 10,000 pounds per square inch. Would a direct tensile stress of 12,000 pounds per square inch combined with a shearing stress of 7,000 pounds per square inch be allowable?

Ans. No, since the maximum resultant tensile stress of 15,226 pounds per square inch exceeds the limit in tension.

2. If the unit tensile stress in Problem 1 were 8,000 pounds per square inch and the unit shearing stress were 9,000 pounds per square inch, what combined stress would determine the safety?

Ans. The combined shearing stress, which is 9,849 pounds per square inch, is only a little below the limit.

3. If the allowable shearing stress is two-thirds of the allowable tensile stress, for what ratios of direct shear to direct tension will each govern the design?

Ans. If s_s is less than $\frac{\sqrt{3}}{2}s_t$, the combined tensile stress governs the design. If greater, the combined shearing stress governs.

181. Elastic Hysteresis.—In elastic bodies subjected to stress, the deformation lags behind the applied force. If a load is applied to a body in tension, it will stretch quickly for a considerable amount but will continue to stretch a little more for some time. If the load is reduced the body will shorten in a similar way. When a steel rod is stretched in an ordinary testing machine the load is applied by means of the screws until the beam is balanced at the desired load. If the machine be then stopped, the beam slowly *falls* due to the increased stretch. This may take several minutes. If the machine be run a little again so as to lift the beam, it will come down much slower the next time, if it comes down at all. If the stress be near the yield point, the beam will drop more quickly and it may have to be lifted several times before it can be made to permanently support the load.

A similar effect is produced when the load is decreased. If the poise of the testing machine be set at a given load and machine turned until the beam is balanced and then stopped, the beam will *rise* slowly. This indicates that the rod is continuing to shorten.

Table XVII shows the first of these effects in the case of soft steel. In making the test the machine was run rapidly and then

stopped. The reading of the load was then taken. After one minute the poise was moved back until the beam again balanced. For instance, the machine was balanced at 25,000 pounds per square inch. After one minute the balance was 24,000 pounds per square inch.

TABLE XVII.—TEST OF SOFT STEEL IN TENSION

Area of section, 0.600 square inch

Total load		Unit stress per square inch		Elongation when machine stopped	
When machine stopped	After one minute	When machine stopped	After one minute	In 8 inches	Unit
Pounds	Pounds	Pounds	Pounds	Inches	Inch
30	30	50	50	0.0	0.0
6,150	6,000	10,250	10,000	0.00270	0.00034
9,000	8,650	15,000	14,420	0.00405	0.00051
12,200	11,800	20,330	19,670	0.00550	0.00069
15,000	14,400	25,000	24,000	0.00675	0.00084
18,000	17,100	30,000	28,500	0.0082	0.00102
19,200	18,250	32,000	30,420	0.0085	0.00106
19,800	18,600	33,000	31,000	0.0087	0.00109
20,400	19,150	34,000	31,920	0.0092	0.00115
21,000	19,200	35,000	32,000	0.0121	0.00151
20,600	19,350	34,330	32,250	0.0450	0.00562
20,600	20,000	34,330	33,330	0.0530	0.00662
21,000	19,000	35,000	31,670	0.0733	0.00916
21,000	19,200	35,000	32,000	0.1740	0.02175
21,600	20,900	36,000	34,830	0.2121	0.02651
22,800	21,750	38,000	36,250	0.2393	0.02991
24,000	22,900	40,000	38,160	0.2773	0.03466
26,400	25,050	44,000	41,750	0.3873	0.04841
28,800	26,900	48,000	44,830	0.5506	0.06882
31,200	29,050	52,000	48,520	0.83	0.104
32,400	30,600	54,000	51,000	1.30	0.162
32,800	30,850	54,670	51,420	1.62	0.202
32,850	31,200	54,750	52,000	1.89	0.236
32,750	30,950	54,580	51,580	2.08	0.260
32,600	30,900	54,330	51,500	2.52	0.315
23,400	....	39,000	....	2.98	0.372

Broke at 23,400 pounds. The area of the neck was 0.196 square inch.

The curves of Fig. 209 were plotted from the results of Table XVII. The apparent ultimate strength of this steel is 54,750 pounds per square inch. The actual quiescent load which it would permanently support is less than 52,000 pounds per square inch.

From this table and the curves it is evident that the speed of the test is an important factor in determining the apparent strength of a ductile material. Also the modulus of elasticity, when taken rapidly, is larger than if taken slowly.

If the material be once stretched to nearly the yield point and the load then removed, a repetition of the load will show less lag of the deformation.

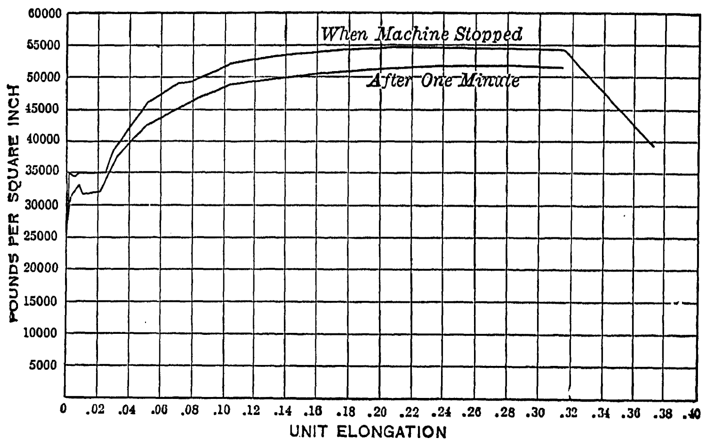


FIG. 209.—Effect of time on stress-strain diagram.

Table XVIII gives a part of the data for the test of a rod taken from the same bar as that of Table XVII. The rod was raised to 42,000 pounds per square inch producing an elongation of 0.3 inch in the gage length of 8 inches. It was then brought back to 50 pounds per square inch. The unit stress was calculated from the original area of 0.6 square inch and the unit elongation from the original gage length.

It will be noticed that the effect of time is much less than in Table XVII. For instance, in Table XVII the unit stress falls from 35,000 to 32,000 in one minute while in Table XVIII it falls from 35,000 to 34,830 in the same time.

Table XVIII shows also the increase of stress after a short interval when the load has been decreased. For instance, when

Table XVIII.—Repeated Test of Soft Steel

Total load		Unit stress per square inch		Elongation when machine stopped	
When machine stopped	After one minute	When machine stopped	After one minute	In 8 inches	Unit
Pounds	Pounds	Pounds	Pounds	Inch	Inch
30	30	50	50	0	0
3,000	3,000	5,000	5,000	0.00105	0.00013
6,000	6,000	10,000	10,000	0.00260	0.00032
9,000	8,900	15,000	14,830	0.00435	0.00054
12,000	11,900	20,000	19,830	0.00610	0.00076
15,000	14,950	25,000	24,920	0.00785	0.00098
18,000	17,825	30,000	29,710	0.00965	0.00121
21,000	20,900	35,000	34,830	0.01160	0.00145
24,000	23,600	40,000	39,330	0.01355	0.00169
24,600	24,200	41,000	40,330	0.01540	0.00192
25,200	24,550	42,000	40,920	0.01765	0.00221
24,000	23,900	40,000	39,830	0.01735	0.00217
21,000	21,150	35,000	35,250	0.01580	0.00197
18,000	18,100	30,000	30,170	0.01420	0.00177
15,000	15,075	25,000	25,125	0.01230	0.00154
12,000	12,225	20,000	20,375	0.01080	0.00135
9,000	9,300	15,000	15,500	0.00900	0.00112
6,000	6,225	10,000	10,375	0.00730	0.00091
3,000	3,125	5,000	5,210	0.00520	0.00065
30	120	50	200	0.00350	
30	30	50	50	0.00330	0.00041

the load was dropped to 20,000 pounds per square inch, after one minute it was found to have increased to 20,375 pounds per square inch due to the continued shortening of the test piece.

Fig. 210 shows the behavior of this test piece. Curve I is the stress-strain diagram up to the yield point at a little under 35,000 pounds per square inch. This is practically the same as the results of Table XVII. The interval from *B* to *C*, representing unit elongation of 0.0360, is omitted. The curve is then drawn from *C* to *D*. From *D* the load is lowered to 50 pounds per square inch. Table XVIII begins at this point at unit elongation of 0.0375 which corresponds with 0 of the table. From 5,000 to 40,000 pounds per square inch the curve is nearly a straight line. At 40,000 pounds there is a rapid change in slope and the effect

of time is very greatly increased. There is practically a new yield point at 40,000 pounds per square inch, whereas the original yield point was under 35,000 pounds per square inch. It will be seen from Table XVII that when the apparent unit stress was 42,000 pounds per square inch it dropped to about 40,000 pounds after one minute. The rod of Table XVIII was raised to 42,000 pounds and then lowered and the new yield point on the second application of load is now raised to the load which it would *permanently* support with the elongation originally produced by 42,000 pounds per square inch.

When steel or wrought iron is stressed beyond the yield point, the yield point at the next application of load is found at the permanent stress previously reached.

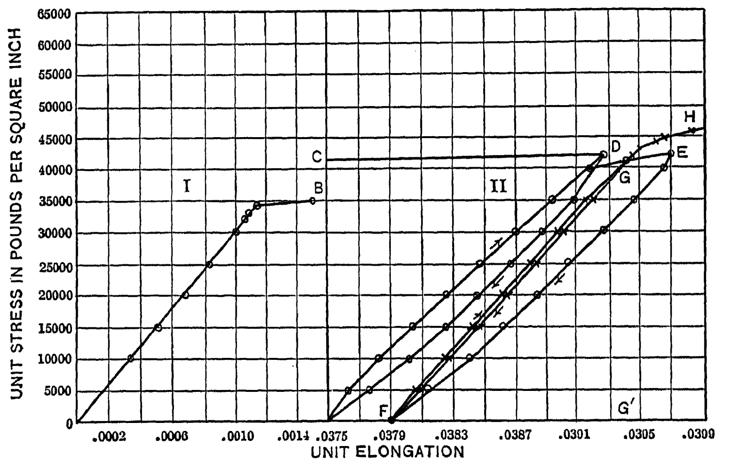


FIG. 210.—Stress-strain cycles.

The elastic limit is also raised. On the other hand, if the *yield point* in *tension* is *raised* by straining the bar beyond its original yield point, the *yield point* in *compression* is *lowered* so that the length of the *interval between these two yield points remains nearly constant.*

The descending curves, from D downward and again from E to F, are concave toward the left while the ascending curves are concave toward the right. This leaves an area between the ascending and descending curves which represents the energy lost in the cycle, and may be called the *loop of elastic hysteresis.*

At F the bar was allowed to rest for 40 hours without load.

It was then raised to G at 40,000 pounds per square inch and again lowered to 50 pounds per square inch. It will be seen that the hysteresis loop has a small area, and that both the ascending and descending curves are nearly straight. Also, there is no temporary set.

Fig. 211 shows some of the results of similar experiments at the Watertown Arsenal ("Tests of Metals," 1886, Part 2, pages 1571–1617). These tests were made on eye-bars about 25 feet long. The gage length was 260 inches which made it possible to measure the elongation with great relative accuracy.

Table XIX represents the first part of the test. The initial load was 1,000 pounds per square inch. After each load the

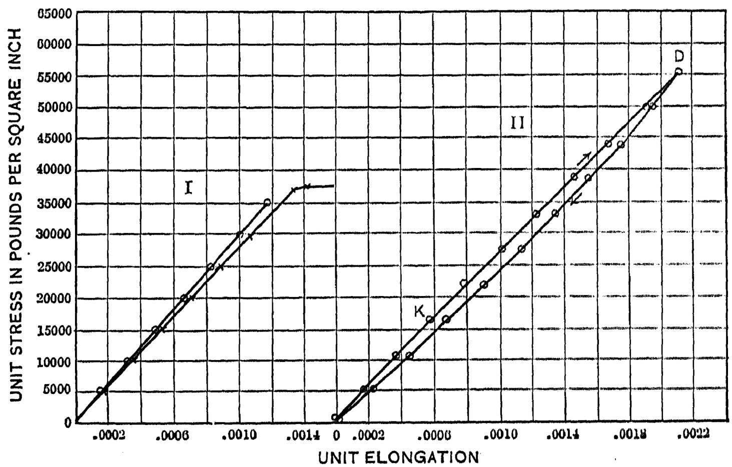


FIG. 211.—Watertown Arsenal test.

machine was reversed to the initial load and a reading taken for set. Under the heading "Unit elongation" the table gives the values obtained by subtracting the original reading at the initial load from the reading at the given load. Under the heading of "Net unit elongation" the table gives the values obtained by subtracting the *set at initial load following a given reading* from the elongation at that load. Curve I of Fig. 211 shows both elongations. The line of greatest slope drawn through the circles represents the net unit elongation. The values of the modulus of elasticity as obtained by the two methods are quite different.

After the load was removed the bar was again tested. The results of one cycle are given in Table XX and by curve II of Fig.

TABLE XIX.—WATERTOWN TESTS OF STEEL EYE-BAR

Total load	Unit stress per square inch	Elongation		Set under initial load	Net unit elongation	E
		In 260 inches	Unit			
Pounds	Pounds	Inches	Inch	Inches	Inch	
5,250	1,000	0.0	0.0			
26,250	5,000	0.0456	0.000175	0.0065	0.000150	26,670,000
52,500	10,000	0.0915	0.000352	0.0085	0.000319	28,210,000
78,750	15,000	0.1369	0.000526	0.0089	0.000485	28,860,000
105,000	20,000	0.1815	0.000698	0.0096	0.000661	28,750,000
131,250	25,000	0.2264	0.000871	0.0101	0.000832	28,850,000
157,500	30,000	0.2720	0.001046	0.0109	0.001004	28,880,000
183,750	35,000	0.3194	0.001229	0.0147	0.001172	29,010,000
196,000	37,330	0.3459	0.001330			
198,000	37,710	0.3700	0.001423			
200,000	38,090	0.5665	0.002179			
204,750	39,000	1.07	0.0041			
210,000	40,000	2.35	0.0090			
369,000	70,286	......	....	30.42		

211. The unit elongation is calculated from the original length of 290.4 inches. Readings were taken at a given load and again after an interval of three minutes with the same load. The results agree with those of Table XVIII. With increasing loads the unit elongation continued to increase and with decreasing loads it continued to decrease. The difference for a three-minute interval was never more than ½ of 1 per cent. of the total elongation.

These tests show that there is a relatively large hysteresis the first time the load is applied and much less hysteresis when the loading is repeated. There is also some set which slowly vanishes.

The form of the ascending curve at the second application of the load depends largely upon the length of time which has elapsed after the first load was removed. If the second loading is applied before most of the temporary set has vanished the curve will be very steep at first and will have an apparent elastic limit at a low stress. This is seen in Fig. 210 where there is a change in slope at 5,000 pounds per square inch, and then practically a straight line up to D. The curve starting from F has no such bend because the bar rested before this load was applied.

TABLE XX.—WATERTOWN TEST OF STEEL EYE-BAR REPEATED

Area, 4.70 square inches. Gage length, 290.4 inches.

Bar previously stretched from 260 inches by a load of 369,000 pounds

Total load	Unit stress per square inch	Elongation in 290.4 inches		Immediate unit elongation
		Immediate	After three minutes	
Pounds	Pounds	Inch	Inch	Inch
5,250	1,117	0		
26,250	5,585	0.0503	0.0504	0.000173
52,500	11,170	0.1093	0.1099	0.000376
78,750	16,755	0.1685	0.1690	0.000582
105,000	22,340	0.2299	0.2309	0.000792
131,250	27,925	0.2922	0.2935	0.001006
157,500	33,510	0.3562	0.3575	0.001226
183,750	39,095	0.4209	0.4222	0.001449
210,000	44,680	0.4868	0.4885	0.001676
236,250	50,265	0.5533	0.5549	0.001905
262,500	55,850	0.6209	0.6230	0.002148
236,250	50,265	0.5660	0.5659	0.001949
210,000	44,680	0.5082	0.5080	0.001750
183,750	39,095	0.4491	0.4489	0.001546
157,500	33,510	0.3886	0.3880	0.001338
131,250	27,925	0.3262	0.3252	0.001123
105,000	22,340	0.2631	0.2620	0.000906
78,750	16,755	0.1980	0.1971	0.000682
52,500	11,170	0.1331	0.1319	0.000455
26,250	5,585	0.0655	0.0627	0.000225
5,250	1,117	0.0048	0.0030	0.000016
	Bar rested 15 hours under initial load.			
5,250	......	−0.0054		

182. Failure under Repeated Stress.—There is a considerable area between the ascending and descending portions of the stress-strain diagram in Figs. 210 and 211. The greater the limits of stress the greater this area. This inclosed area is a measure of the work expended in stretching the bar which is not recovered as mechanical work when the load is released.

Since energy is lost in a cycle of this kind it is natural to expect that a great number of repetitions of stress would cause failure

at a maximum stress less than the ultimate strength of the material. The experiments of Wöhler and others show that this is the case.*

When the stress varied from zero to a maximum it was found that if this maximum was less than one-half the ultimate strength the piece would fail under a great number of repetitions of load. If a steel bar having an ultimate strength of 60,000 pounds per square inch is loaded from 0 to 40,000 pounds it will probably break after a few thousand applications. If loaded from 0 to 35,000 it will last much longer. If from 0 to 30,000 it *may* fail after several million repetitions. If loaded from 0 to 25,000 it will last indefinitely.†

The smaller the range of stress the higher the maximum may be without failure under an indefinite number of repetitions. Steel having an ultimate strength of 60,000 pounds per square inch will stand a stress varying from 25,000 to 40,000 pounds per square inch without failure.

When the stress changes from tension to compression the maximum stress is still less than for the case of one kind of stress. Experiments show that when a bar is tested in one direction its elastic limit in the other is lowered, so that the raising of the elastic limit which occurs when a bar of ductile material is overstrained in one direction is lost when the reverse stress is applied. The experiments of Wöhler show that steel having an ultimate strength of 60,000 pounds per square inch when tested in tension will fail under a stress which changes from 16,000 compression to 16,000 pounds tension. If the stress changes from 14,000 tension to 14,000 compression the piece will probably stand an indefinite number of repetitions.

183. Design for Varying Stresses.—A number of methods have been proposed for designing members subjected to repeated stresses. This may be done by lowering the allowable unit stress or adding a suitable increment to the applied load.

* See GOODMAN'S "Mechanics Applied to Engineering," under the head "Wöhler's Experiments." UNWIN'S "The Testing of Materials of Construction," pages 356–394, gives an excellent discussion of this subject. Also see paper by HENRY B. SEAMAN, *Transactions of the American Society of Civil Engineers*, Vol. XLVI (1899), pages 141–150, and discussion on pages 166–257.

† See paper by J. H. SMITH entitled "Some Experiments on Fatigue of Metals," *The Journal of the Iron and Steel Institute*, 1910, Vol. II, pages 246–318.

The formula of Launhardt is an empirical formula based on Wöhler's experiments, which until recently was considerably used for calculating the allowable working stress for varying loads. This formula contains a factor depending upon the ratio of the ultimate static strength to the ultimate repetition strength when the load varies from 0 to the maximum. If we take this ratio as 2 which coincides reasonably well with the results of the tests, the formula may be written

$$s_v = \frac{s_w}{2}\left(1 + \frac{\text{minimum load}}{\text{maximum load}}\right),$$

where

s_w = static allowable unit stress,
s_v = maximum allowable unit stress with varying load.

When the minimum load is 0, $s_v = \dfrac{s_w}{2}$.

When the minimum load equals the maximum load, $s_v = s_w$.

Problems

1. If the allowable unit stress for a given steel for a static load is 15,000 pounds per square inch, what is the maximum allowable unit load and the required area of cross-section when the load varies from 20,000 to 30,000?
Ans. 12,500 pounds per square inch, 2.4 square inches.

2. Find the area of cross-section to carry safely a load which varies from 120,000 to 360,000 pounds if the allowable static unit stress is 15,000 pounds per square inch. *Ans.* 36 square inches.

Launhardt's formula applies to stresses in one direction only. Goodman* recommends a simple rule which is easy to remember and convenient to apply. *Add to the maximum load the difference between the maximum and minimum load and treat the sum as a static load.*

Problems

3. Solve Problem 1 by Goodman's "dynamic" rule.
Ans. 2.67 square inches.

4. What is the cross-section required to carry a load which varies from 30,000 pounds compression to 60,000 pounds tension if the allowable static unit stress is 12,000 pounds per square inch? *Ans.* 12.5 square inches.

5. A shaft is supported between bearings 4 feet apart and carries a load of 600 pounds at the middle. If the allowable static unit stress is 12,000

* GOODMAN'S "Mechanics Applied to Engineering," page 535. For a discussion of the various formulas see JOHNSON'S "Materials of Construction," Art. 389.

pounds, what is the minimum diameter of the shaft to allow for the alternate tension and compression as the shaft rolls over?

If the shaft makes 100 revolutions per minute in what time will the stress reverse one million times?

Prof. O. H. Basquin* has shown that the behavior of metals under repeated stress may be expressed by the exponential formula,

$$S = KN^q$$

where N is the number of repetitions, K and q are constants which must be determined experimentally from endurance tests of each material, and S is the unit stress at which the material will fail with N repetitions of stress. This formula applies to stresses from the yield point down to a little below the elastic limit as ordinarily determined.

184. Crystallization under Repeated Stress.—When steel fails under repeated applications of load, the fracture has a crystalline appearance. For this reason it was long thought that repeated stresses cause the formation of crystals in the steel. Microscopic† examination shows that all steel is crystalline and that crystals do not form at atmospheric temperatures. The crystalline appearance of the fracture is due to the fact that the fracture has taken place across the crystals of the steel.

* "The Exponential Law of Endurance Tests," *Proceedings of the American Society for Testing Materials*, 1910, page 625. Prof. H. F. Moore and B. F. Seely have given a complete discussion of repeated stresses in the *Proceedings of the Society for Testing Materials*, 1915, pages 438–460.

† Read Chapter XI of ROSENHAIN'S "Introduction to Physical Metallurgy," on "The Effect of Strain on the Structure of Metals," for the most recent discoveries and theories on this subject.

Read HOWE'S "Metallography of Steel and Cast Iron."

CHAPTER XVIII

CURVED BEAMS AND HOOKS

185. Stresses in Curved Beams.—Fig. 212 represents a portion of a curved beam between two planes AB and CD, each of which pass through the center of curvature of the beam. The plane AB, at the left end, is regarded as fixed while the plane CD, at the right end, is rotated through an angle θ to the position $C'D'$ when the beam is bent. The unit stresses in a beam of this kind *do not vary* directly as the distance from the neutral axis, because

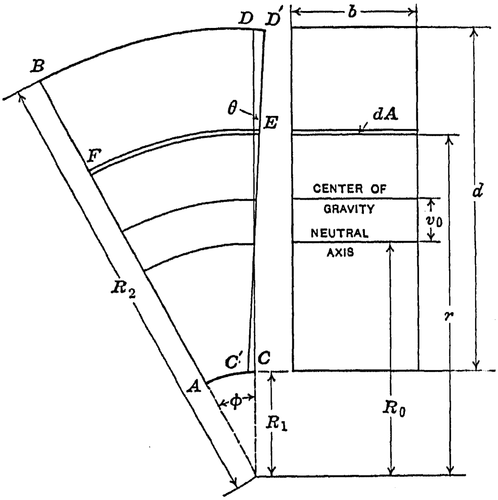


FIG. 212.—Curved beam of rectangular section.

the length of the filaments are not the same. If the neutral axis were midway between C and D the elongation DD' would be equal to the compression CC' but the *unit elongation* at the top would be less than the *unit compression* at the bottom, since the original length BD is greater than AC.

The length of any filament such as EF is proportional to its distance from the center of curvature, so that the unit deformation and the unit stress vary as the quotient of the distance

from the neutral axis divided by the distance from the center of curvature.

In Fig. 212, R_1 is the inside radius, R_2 is the outside radius, R_0 is the radius of the neutral surface, r is the radius of any filament and v_0 is the distance of the neutral axis from the center of gravity of the cross-section. The angle at the center of curvature subtended by the portion of the beam is ϕ, so that the original length of any filament is $r\phi$. The angle through which the plane CD is turned when the beam is bent is θ. If the assumption that a cross-section of a beam remains plane when the beam is bent *be valid for curved as well as for straight beams*, the deformation of a filament at a distance $r - R_0$ from the neutral axis is $(r - R_0)\theta$ and

$$\text{Unit deformation} = \frac{(r - R_0)\theta}{r\phi}. \qquad (1)$$

$$\text{Unit stress} = s = \frac{E(r - R_0)\theta}{r\phi} = k\left(1 - \frac{R_0}{r}\right). \qquad (2)$$

At the innermost fibers

$$S_1 = k\left(1 - \frac{R_0}{R_1}\right). \qquad (3)$$

At the extreme outer fibers

$$S_2 = k\left(1 - \frac{R_0}{R_2}\right). \qquad (4)$$

The location of the neutral axis is found by means of the condition that the total stress across any section is zero.

$$\text{Stress on element of area } dA = k\left(1 - \frac{R_0}{r}\right)dA; \qquad (5)$$

$$\text{Total stress} = k\int_{R_1}^{R_2}\left(1 - \frac{R_0}{r}\right)dA = 0. \qquad (6)$$

$$A = R_0\int_{R_1}^{R_2}\frac{dA}{r};\; R_0 = \frac{A}{\displaystyle\int_{R_1}^{R_2}\frac{dA}{r}}. \qquad (7)$$

The resisting moment of the stress on the area dA is the product of this stress multiplied by distance $r - R_0$.

$$M = k\int_{R_1}^{R_2}\frac{(r - R_0)^2}{r}\,dA. \qquad (8)$$

The values given by equations (7) and (8) depend upon the form of the section, that is, upon the value of dA as a function of r.

To find the unit stress at any point in terms of the moment, eliminate k between equations (2) and (8), and to find S at the inner or outer fibers, use R_1 or R_2 for r in the equation thus obtained.

186. Curved Beams of Rectangular Section.—For a rectangular beam of unit width, $dA = dr$ and $A = R_2 - R_1 = d$, where d is the depth. From equation (2) of Article 185,

$$R_0 = \frac{A}{\int_{R_1}^{R_2} \frac{dr}{r}} = \frac{A}{\log \frac{R_2}{R_1}} = \frac{d}{\log \frac{R_1 + d}{R_1}}. \tag{1}$$

$$r_0 = \frac{R_2 + R_1}{2} - R_0 = R_1 + \frac{d}{2} - R_0; \tag{2}$$

Example

In a beam of rectangular section the inner radius is 4 inches and the outer radius is 8 inches. Find the distance of the neutral axis from the center of gravity of the section.

$$d = 4 \text{ inches}; \quad \frac{R_2}{R_1} = 2;$$

$$R_0 = \frac{4}{\log_e 2} = \frac{4}{0.69315} = 5.771 \text{ inches}.$$

$$r_0 = 6 - 5.771 = 0.229.$$

$$\frac{r_0}{d} = 0.0572.$$

In a rectangular beam of depth equal to the radius of the inner surface, the neutral axis is shifted toward the center of curvature 0.0572 of the depth, or nearly 6 per cent.

To find the resisting moment, substitute dr for dA in equation (8) of the preceding article.

$$M = k\int_{R_1}^{R_2}\left(r - 2R_0 + \frac{R_0^2}{r}\right)dr = k\left[\frac{r^2}{2} - 2R_0 r + R_0^2 \log r\right]_{R_1}^{R_2} \tag{3}$$

$$M = k\left(\frac{R_2^2 - R_1^2}{2} - 2R_0(R_2 - R_1) + R_0^2 \log \frac{R_2}{R}\right). \tag{4}$$

Substituting the value of R_0 from equation (1),

$$M = k\left(\frac{R_2^2 - R_1^2}{2} - \frac{2(R_2 - R_1)^2}{\log \frac{R_2}{R_1}} + \frac{(R_2 - R_1)^2}{\log \frac{R_2}{R_1}}\right); \tag{5}$$

$$M = k\left(\frac{R_2^2 - R_1^2}{2} - \frac{(R_2 - R_1)^2}{\log \frac{R_2}{R_1}}\right) = kd\left(\frac{R_2 + R_1}{2} - \frac{d}{\log \frac{R_2}{R_1}}\right); \tag{6}$$

$$M = kd\left(\frac{R_2 + R_1}{2} - R_0\right) = kv_0 d. \tag{7}$$

For a rectangular section of width b instead of unity, multiply by b,

$$M = kv_0bd. \tag{8}$$

Equation (7) may be derived more quickly by taking moments with respect to an axis through the center of curvature. The moment arm of the stress on an element dA is now equal to r, and

$$M = k\int\left(1 - \frac{R_0}{r}\right)rdA = k\int(r - R_0)dr,$$

for a rectangular section of unit breadth.

$$M = k\left[\frac{r^2}{2} - R_0r\right]_{R_1}^{R_2} = k\left(\frac{R_2^2 - R_1^2}{2} - R_0(R_2 - R_1)\right); \tag{7}$$

$$M = kd\left(\frac{R_2 + R_1}{2} - R_0\right) = kv_0d.$$

For a beam of breadth b, equation (7) becomes

$$M = kv_0bd \tag{8}$$

Eliminating k between (8) and equation (2) of the preceding article,

$$S = \frac{M\left(1 - \frac{R_0}{r}\right).}{bv_0d} \tag{9}$$

In the above example, to find the unit stress at the inner fibers, where $r = R_1 = 4$ inches,

$$S_1 = \frac{M\left(1 - \frac{5.771}{4}\right)}{4 \times 0.229} = -\frac{0.4427\ M.}{0.916} = -0.483\ M.$$

For a straight beam 1 inch wide and 4 inches deep, the unit stress in the extreme fibers is 0.375 M. The unit stress at the inner fibers of a rectangular beam for which the outer radius is twice the inner radius is $0.483 \div 0.375 = 1.288$ times as great as the stress in the extreme fibers of a straight beam of the same section.

Table XXI below gives the displacement of the neutral surface and the ratio of the unit stresses in the extreme inner concave surface and the extreme outer convex surface of a curved beam of rectangular section to the unit stresses in the extreme fibers of a straight beam of the same section.

Fig. 213 is plotted with $\frac{d}{R_1}$ as abscissa. The upper curve is

TABLE XXI.—DISPLACEMENT OF NEUTRAL SURFACE AND EXTREME FIBER STRESSES IN CURVED BEAMS OF RECTANGULAR SECTION

Ratio of depth to inner radius, $\frac{d}{R_1}$	Distance of neutral axis from center in terms of depth, $\frac{v_0}{d}$	Unit stresses in extreme fibers compared with straight beams of same section	
		Concave	Convex
0.50	0.0326	1.153	0.875
1.00	0.0572	1.288	0.811
1.50	0.0753	1.409	0.764
2.00	0.0897	1.523	0.726
3.00	0.1120	1.733	0.682
4.00	0.1287	1.923	0.652

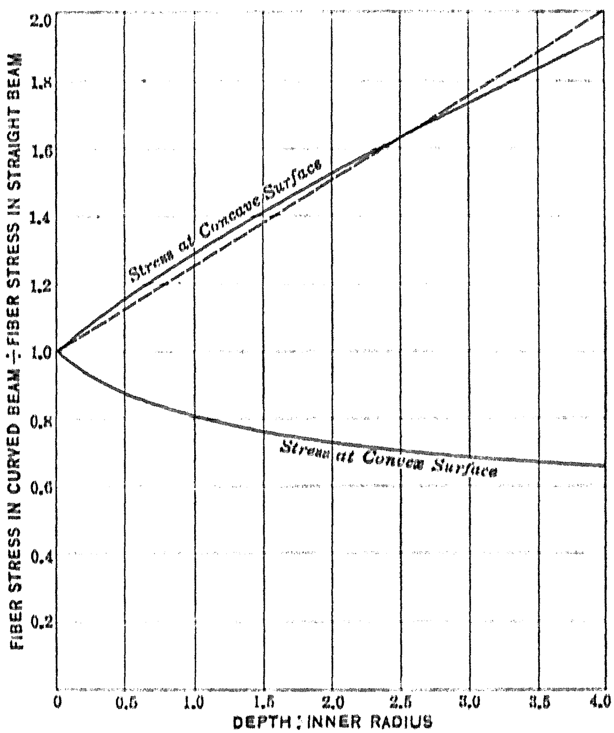


FIG. 213.—Stress at outer fibers of curved beam of rectangular section.

the ratio of the unit stress at the concave surface of the curved beam to the unit stress in a straight beam of the same section. The lower curve is a similar ratio for the unit stress at the convex surface of the curved beam.

Example

By means of the curves of Fig. 213 find the maximum unit stress in a rectangular beam 4 inches wide with inside radius 2 inches and outside radius 7 inches, due to a bending moment of 150,000 inch-pounds.

The unit stress in the extreme fibers of a straight beam of this section, due to this moment, is 9,000 pounds per square inch. From the curve the ratio is 1.63 so that the unit stress in the inner fibers is $9{,}000 \times 1.63 = 14{,}670$ pounds per square inch.

Problems

1. Verify Table XXI for $\frac{d}{R_1} = 3$.

2. By means of Table XXI or Fig. 213 find the unit stress in the extreme fibers and displacement of the neutral axis in the case of a beam of rectangular section 6 inches wide and 7 inches deep, curved with inner radius 2 inches, due to a bending moment of 200,000 inch-pounds.

The stresses at the concave surface are numerically the greatest and are the most important, except in the case of a curved cast-iron beam with the convex surface in tension. The dotted straight line of Fig. 213 with a slope of 0.25 differs little from the actual stress ratio for the concave surface. The equation of this line is

$$y = 1 + 0.25\frac{d}{R_1} \tag{10}$$

where y is the ratio of the unit stress in the curved beam at the concave surface to the unit stress in a straight beam of the same section. Based on this line, the unit stress at the concave surface of a beam is given by the *approximate formula*

$$S_1 = \frac{6\,M}{bd^2}\left(1 + 0.25\,\frac{d}{R_1}\right). \tag{11}$$

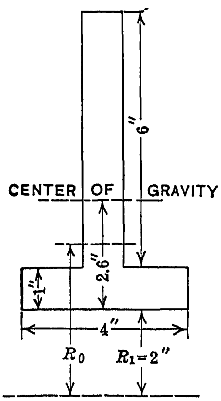


FIG. 214.—T-section.

Problem

3. Find the unit stress in the fibers at the concave surface of a curved beam of inner radius 5 inches, depth 8 inches, and breadth 4 inches, due to a bending moment of 25,600 inch-pounds. Solve by equation (11).

Ans. $S = 600(1 + 0.4) = 840$ pounds per square inch.

187. Beams of T-section.—Fig. 214 shows a T-section. The method of calculation is the same as that used for a rectangle except that two sets of limits are required.

$$R_0 = \frac{A}{4\log\frac{3}{2} + \log\frac{9}{3}} = \frac{10}{4\times 0.40546 + 1.09861} = 3.676 \text{ inches}$$

The center of gravity is 2.6 inches from the inner surface so that

$$v_0 = 2.6 - 1.676 = 0.924 \text{ inch.}$$

With T- or I-sections the relative displacement of the neutral axis is greater than in a rectangular section.

To find the bending stress the integral of the second method of Article 186 must be taken between two sets of limits after multiplying by the breadth. The equation for the unit stress at the concave surface is

$$S_1 = \frac{(R_1 - R_0)\,M}{bR_1\left[\frac{r^2}{2} - R_0 r\right]}, \tag{1}$$

the denominator of which is taken between the limits R_1 and R_2 in the case of a rectangular section. With the T-section of Fig. 214 the breadth b is 4 inches for the limits $r = 2$ to $r = 3$, and the breadth b is 1 inch for the limits $r = 3$ to $r = 9$, so that equation (1) becomes

$$S_1 = \frac{(2 - 3.676)\,M}{2\left[2\left(3^2 - 2^2\right) + \frac{9^2 - 3^2}{2} - 4 \times 3.676 - 6 \times 3.676\right]}.$$

$$S_1 = -\frac{1.676\,M}{2(46 - 36.76)} = 0.0908\ M.$$

At the convex surface,

$$S_2 = \frac{5.324\,M}{9 \times 9.24} = 0.064M.$$

Cast-iron beams are frequently made of T-section and used with the stem in compression so as to make the compressive stress greater than the tensile stress. In a straight beam of the section of Fig. 214 the unit stresses in the outer fibers would be in the ratio of 44 : 26. With a curved beam the advantage of using a T-section is not so great.

Problem

Find the unit stresses in the extreme fibers of a rectangular beam 2 inches wide and 5 inches deep with $R_1 = 2$ inches. Compare the results with the T-section of the above example.

Ans. $S_1 = 0.212\ M.$
$S_2 = 0.089\ M.$

188. Curved Beam of Circular Section.—
To find R_0,

$$dA = -2\,a \sin\theta\, dr;$$
$$r = c + a \cos\theta;$$
$$dr = -a \sin\theta d\theta;$$
$$dA = 2\,a^2 \sin^2\theta d\theta, \tag{1}$$

where c is the radius from the center of curvature to the center of the circle, and a is the radius of the circle.

$$\frac{dA}{r} = \frac{2\,a^2 \sin^2 \theta d\theta}{c + a \cos \theta}. \tag{2}$$

$$\frac{dA}{r} = \left(-\,2\,a \cos \theta + 2\,c - \frac{2\,(c^2 - a^2)}{c + a \cos \theta}\right) d\theta. \tag{3}$$

$$\int \frac{dA}{r} = \left[-\,2\,a \sin \theta + 2\,c\theta + 2\sqrt{c^2 - a^2}\left(\sin^{-1} \frac{a + c \cos \theta}{c + a \cos \theta}\right)\right]_0^{\pi} \tag{4}$$

$$= 2\,c\pi + 2\sqrt{c^2 - a^2}\left(\sin^{-1} \frac{a - c}{c - a} - \sin^{-1} \frac{a + c}{c + a}\right) \tag{5}$$

$$= 2\,c\pi - 2\,\pi\sqrt{c^2 - a^2} = 2\,\pi\,(c - \sqrt{c^2 - a^2}). \tag{6}$$

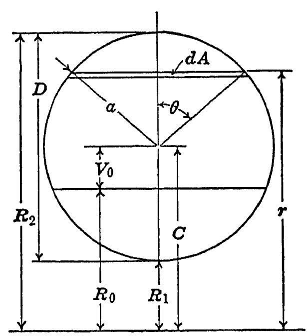


FIG. 215.—Circular section.

$$R_0 = \frac{\pi a^2}{2\,\pi(c - \sqrt{c^2 - a^2})} = \frac{c + \sqrt{c^2 - a^2}}{2}. \tag{7}$$

$$\text{Since } c = \frac{R_2 + R_1}{2} \text{ and } a = \frac{R_2 - R_1}{2}.$$

$$R_0 = \frac{R_1 + 2\sqrt{R_1 R_2} + R_2}{4} = \frac{(\sqrt{R_1} + \sqrt{R_2})^2}{4}. \tag{8}$$

To find the resisting moment in a curved beam of circular section with respect to an *axis through the center of curvature*,

$$M = k \int \left(1 - \frac{R_0}{r}\right) r dA = k \int (r - R_0) a^2 \sin^2 \theta d\theta; \tag{9}$$

$$M = k \int \left((c - R_0)\,2\,a^2 \sin^2 \theta + 2\,a^3 \sin^2 \theta \cos \theta\right) d\theta; \tag{10}$$

$$M = ka^2\left[(c - R_0)\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{2}{3}a\sin^3\theta\right]_0^\pi \tag{11}$$

$$M = ka^2\pi(c - R_0). \tag{12}$$

The moment in these equations is taken with respect to an axis through the center of curvature, but this is the same as the mo-

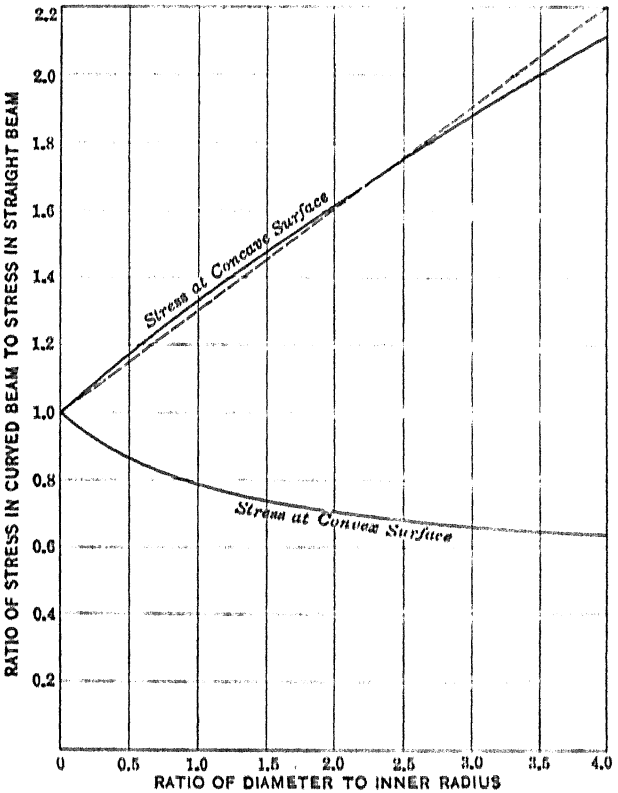


FIG. 216.—Unit stress in outer fibers of curved beam of circular section.

ment with respect the neutral axis provided the total compression is equal to the total tension, in which case the tension and compression form a couple.

At the concave surface where $r = R_1$,

$$S_1 = k\left(1 - \frac{R_0}{R_1}\right).$$

Combining with equation (12),

$$S_1 = \frac{\left(1 - \frac{R_0}{R_1}\right) M}{\pi a^2 (c - R_0)}. \tag{13}$$

Substituting for R_0 and c in equation (13),

$$S_1 = \frac{(3\,R_1 - 2\sqrt{R_1 R_2} - R_2) M}{\pi R_1 a^2 (R_1 - 2\sqrt{R_1 R_2} + R_2)}. \tag{14}$$

At the convex surface where $r = R_2$,

$$S_2 = \frac{(3\,R^2 - 2\sqrt{R_1 R_2} - R_1) M}{\pi R_2 a^2 (R_1 - 2\sqrt{R_1 R_2} + R_2)}. \tag{15}$$

Example

The inner radius is 4 inches and the outer radius is 9 inches. Find the displacement of the neutral axis from the center of gravity of the section, and find the ratio of the unit stress in the extreme fibers to the unit stress in a straight beam of the same cross-section.

$$R_0 = \frac{4 + 12 + 9}{4} = 6.25 \text{ inches.}$$

$$v_0 = 6.5 - 6.25 = 0.25 \text{ inch.}$$

The relative displacement, in terms of the radius is $0.25 \div 2.5 = 0.10$ the displacement of the neutral axis is one-tenth of the radius.

Substituting in (14),

$$S_1 = \frac{(12 - 12 - 9)}{4\,\pi a^2 (4 - 12 + 9)} = -\frac{9\,M}{4\,\pi a^2} = -\frac{2.25\,M}{\pi a^2}.$$

Substituting in (15),

$$S_2 = \frac{(27 - 12 - 4)\,M}{9\,\pi a^2 (4 - 12 + 9)} = \frac{11\,M}{9\,\pi a^2} = \frac{1.22\,M}{\pi a^2}.$$

In a straight beam, 5 inches in diameter,

$$S = \frac{M}{\frac{\pi a^3}{4}} = \frac{4\,M}{\frac{5\,\pi a^2}{2}} = \frac{1.6\,M}{\pi a^2},$$

which is a little less than one-half the sum of the unit stresses in concave and convex fibers of the curved beam.

Problems

1. A beam of circular section is 2 inches in diameter, and is curved so that the inner radius is 1 inch. Find the displacement of the neutral surface and the unit stress in the extreme fibers compared with the unit stress in a straight beam of the same section.

Ans. $v_0 = 0.134$ inch; $S_1 = 1.616\,S$; $S_2 = 0.705\,S$.

2. Show that in a beam of circular section the maximum possible displacement of the neutral axis is one-half the radius.

TABLE XXII.—CURVED BEAMS OF CIRCULAR SECTION

Ratio of diameter to inner radius, $\frac{D}{R_1}$	Distance of neutral axis from center in terms of diameter $\frac{v_0}{D}$	Unit stresses compared with stresses in straight beam of same section	
		Concave, $\frac{S_1}{S}$	Convex, $\frac{S_2}{S}$
0.2	0.0114	1.071	0.935
0.4	0.0210	1.142	0.887
0.6	0.0293	1.207	0.841
0.8	0.0365	1.271	0.817
1.0	0.0429	1.332	0.791
1.5	0.0563	1.478	0.741
2.0	0.0670	1.616	0.705
2.5	0.0758	1.748	0.678
3.0	0.0833	1.875	0.656
4.0	0.0955	2.118	0.623

Fig. 216 is plotted from Table XXII. The relative stress at the concave surface does not differ greatly from that represented by the straight line,

$$S_1 = S\left(1 + 0.3\frac{D}{R_1}\right) = \frac{4M}{\pi a^3}\left(1 + 0.3\frac{D}{R_1}\right). \qquad (16)$$

189. Curved Beam of Trapezoidal Section.—In a trapezoidal section let C be the distance from the center of curvature to the point of intersection of the non-parallel edges of the section, and let m be the increase of width of the section per unit distance measured along the radius, Fig. 217. In the figure, C is greater than R_2 and m is negative. C may be less than R_1, in which case m is positive.

$$s = k\left(1 - \frac{R_0}{r}\right). \qquad (1)$$

An element of area is $m(r - C)dr$, and

$$\text{Total stress} = km\int\left(1 - \frac{R_0}{r}\right)(r - C)\,dr, \qquad (2)$$

$$\text{Total stress} = km\left[\frac{r^2}{2} - R_0 r - Cr + CR_0 \log r\right]_{R_1}^{R_2} \qquad (3)$$

Total stress

$$= km\left(\frac{R_2^2 - R_1^2}{2} - R_0(R_2 - R_1) - C(R_2 - R_1) + CR\log\frac{R_2}{R_1}\right). \quad (4)$$

If there is no resultant stress normal to the section, the total stress is zero, and

$$R_0 = \frac{\left(\frac{R_2 + R_1}{2} - C\right)d}{d - C\log_e\frac{R_2}{R_1}}, \quad (5)$$

where $d = R_2 - R_1$.

$$\text{When } C = 0,\ R_0 = \frac{R_2 + R_1}{2},$$

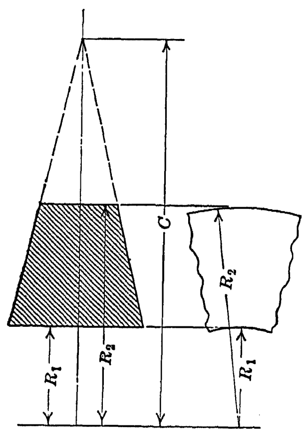


FIG. 217.—Trapezoidal curved beam.

and the neutral axis is midway between the surfaces.

To find the moment of the trapezoidal section with respect to the center of curvature,

$$M = km\int\left(r - R_0\right)\left(r - C\right)dr; \quad (6)$$

$$M = km\int\left(r^2 - (R + C)r + CR_0\right)dr; \quad (7)$$

$$M = km\left[\frac{r^3}{3} - (R_0 + C)\frac{r^2}{2} + CR_0r\right]_{R_1}^{R_2};$$

$$M = kmd\left(\frac{R_2^2 + R_2R_1 + R_1^2}{3} - (R_0 + C)\frac{R_2 + R_1}{2} + CR_0\right). \quad (8)$$

At the concave surface where $r = R_1$,

$$S_1 = \frac{(R_1 - R_0)\, M}{mR_1 d\left(\dfrac{R_2^2 + R_2R_1 + R_1^2}{3} - (R_0 + C)\,\dfrac{R_2 + R_1}{2} + CR_0\right)} \cdot \quad (9)$$

At the convex surface where $r = R_2$, a similar formula gives the fiber stress,

$$S_2 = \frac{(R_2 - R_0)\, M}{mR_2 d\left(\dfrac{R_2^2 + R_2R_1 + R^2}{3} - (R_0 + C)\,\dfrac{R_2 + R_1}{2} + CR_0\right)} \cdot \quad (10)$$

Example

A trapezoidal curved beam is 2 inches wide at the concave surface, 1 inch wide at the convex surface and 2 inches deep. The inner radius is 4 inches. Locate the neutral axis and find the maximum unit stresses in terms of the moment.

$R_1 = 4$ inches, $R_2 = 6$ inches, $C = 8$ inches, $d = 2$ inches, $m = -0.5$.

$$R_0 = \frac{(5 - 8)\,2}{2 - 8 \log_e \dfrac{3}{2}} = \frac{6}{1.2437} = 4.824 \text{ inches.}$$

The center of gravity of the section is 4.889 inches from the center of curvature and $v_0 = 0.065$ inch.

$$S_1 = \frac{(4 - 4.824)\, M}{0.5 \times 4 \times 2\left(\dfrac{36 + 24 + 16}{3} - 12.824 \times 5 + 8 \times 4.824\right)};$$

$$S_1 = \frac{0.824\, M}{4 \times 0.195} = 1.084\, M.$$

$$S_2 = \frac{(6 - 4.824)\, M}{6 \times 0.195} = 1.005\, M.$$

For a straight beam of the same section,

$$S_1 = \frac{12\, M}{13} = 0.923\, M;$$

$$S_2 = \frac{15\, M}{13} = 1.154\, M.$$

It will be noticed that the location of the neutral axis is independent of the slope. The unit stress is inversely proportional to the slope m, that is, it is inversely proportional to the width of the beam. The ratio of the unit stress in the curved beam to the unit stress in a straight beam of the same section is independent of m.

For any given ratio of C to R_1 a set of values might be computed for the ratio of the unit stress in the curved beam to the unit stress in a straight beam of the same section, and a curve might be plotted as was done for the beams of a rectangular and circular section. A set of such curves might be made for various values of the ratio $\dfrac{C}{R_1}$. But since the work of calculating

the center of gravity and section modulus of a trapezoidal section is as laborious as substitution direct in equations (5), (9) and (10) it is not worth while to make these curves.

190. Hooks.—A hook is equivalent to a curved beam subjected to an eccentric tension. As in the case of a short block eccentrically loaded, the total pull P may be replaced by a pull P at the center of gravity of the section and a bending moment Pe, where e is the distance from the center of gravity of the section to the line of the load. The direct tensile stress is $\frac{P}{A}$. If the hook be straight at the section considered, the tensile stress due to bending is $\frac{Pev}{I}$, and at the innermost fibers,

$$S_t = \frac{P}{A}\left(1 + \frac{ec}{r^2}\right). \tag{1}$$

At the outermost fibers, if the hook be straight at the section considered,

$$S_2 = \frac{P}{A}\left(1 - \frac{ec}{r^2}\right). \tag{2}$$

In the case of a hook the eccentricity is always so great that the second term of this last equation is larger than the first so that S_2 is compression.

Equations (1) and (2) afford an approximate method of finding the unit stress in a hook or curved rod subjected to tension or compression, but unless the curvature at the section considered be relatively small there is a considerable error on the side of danger. For more accurate results it is necessary to treat the hook as a curved beam in finding the second term. At the innermost fibers of a hook,

$$S_t = \frac{P}{A} + S_1, \tag{3}$$

where S_1 is the unit stress at the concave surface calculated for a curved beam with the moment equal to Pe. At the outermost fibers,

$$S_o = S_e - \frac{P}{A}. \tag{4}$$

191. Curved Bar of Rectangular Section.—While hooks are not made with a rectangular section, curved bars are sometimes made with sections which are approximately rectangles.

Example

A curved bar of rectangular section is 2 inches wide. At the section farthest from the applied load the inner radius is 3 inches and the outer radius is 6 inches. The load is 3,000 pounds tension and passes through the center of curvature. Find the maximum unit tensile and compressive stress at the most remote section.

$R_1 = 3$ inches; $R_2 = 6$ inches; $d = 3$ inches; $b = 2$ inches.

$M = 3{,}000 \times 4.5 = 13{,}500$ inch-pounds; $A = 6$ square inches;

$$\frac{P}{A} = 500 \text{ lb./in.}^2$$

$$R_0 = \frac{3}{\log_e 2} = \frac{3}{0.69315} = 4.328 \text{ inches.}$$

$$v_0 = 4.500 - 4.328 = 0.172 \text{ inch.}$$

The unit stress at the innermost fiber due to bending, by equation (9) of Article 186 is,

$$S_1 = \frac{(3 - 4.328)\ 13{,}500}{3 \times 2 \times 3 \times 0.172} = 5{,}790 \text{ lb./in.}^2 \text{ tension.}$$

$$S_t = 5{,}790 + 500 = 6{,}290 \text{ lb./in.}^2$$

At the convex surface,

$$S_2 = \frac{(6 - 4.328)\ 13{,}500}{6 \times 3 \times 2 \times 0.172} = 3{,}645 \text{ lb./in.}^2 \text{ compression.}$$

$$S_c = 3{,}638 - 500 = 3{,}145 \text{ lb./in.}^2$$

If the bar were straight and had the same eccentricity of loading, the bending stress in the outer fibers would be

$$S = \frac{13{,}500 \times 6}{2 \times 3^2} = 4{,}500 \text{ pounds per square inch.}$$

The approximate value for the unit bending stress in the curved beam is $4{,}500\ (1 + 0.25) = 5{,}625$ lb./in.^{2}, and approximate $S_t = 5{,}625 + 500 = 6{,}125$ lb./in.^{2}

It will be understood that R_0 in this example is the distance from the center of curvature to the location of the neutral axis, if the beam were subjected to bending only. When the bending is combined with tension the actual surface in which the fibers suffer no deformation is farther from the center. When the line of the load passes through the center of curvature, as in this case, the true neutral surface in a rectangular section passes through the center of the section.

Problems

1. Find the unit stress in the extreme fibers in the above example by means of Table XXI.

2. A curved beam of square section is 2 inches wide. The inner radius is 3 inches and the outer radius is 5 inches. The load is 2,000 pounds and passes through the center of curvature. Find the maximum tensile and compressive stress. *Ans.* $S_t = 7,866$ lb./in.^{2}.

3. Solve Problem 2 by means of the curves of Fig. 213.

192. Hook of Circular Section.—The problem of finding the unit tensile stress at the concave surface of a hook of circular section is solved by means of equation (14) of Article 188. Regarding the tensile stress as positive the complete expression for the concave surface is

$$S_t = \frac{Pe(R_2 + 2\sqrt{R_1R_2} - 3\,R_1)}{\pi a^2 R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} + \frac{P}{\pi a^2}; \tag{1}$$

$$S_t = \frac{P}{\pi a^2}\left(\frac{e(R_2 + 2\sqrt{R_1R_2} - 3\,R_1)}{R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} + 1\right). \tag{2}$$

At the convex surface, from equation (15) of Article 188

$$S_c = \frac{P}{\pi a^2}\left(\frac{e(3\,R_2 - 2\sqrt{R_1R_2} - R_1)}{R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} - 1\right). \tag{3}$$

Example

A hook of circular section is 2 inches in diameter. The inner radius of curvature is 3 inches and the outer radius is 5 inches. The load is 2,000 pounds with the line of its resultant 1 inch inside the concave surface. Find the unit stress in the extreme fibers.

$R_1 = 3$ inches; $R_2 = 5$ inches; $a = 1$ inch; $e = 2$ inches.

$$\frac{P}{\pi a^2} = 636.6 \text{ lb./in.}^2$$

$$S_t = 636.6\left(\frac{2(5 + 2\sqrt{15} - 9)}{3(3 - 2\sqrt{15} + 5)} + 1\right).$$

$$S_t = 636.6\left(\frac{2 \times 3.746}{3 \times 0.254} + 1\right) = 636.6 \times 10.83 = 6,894 \text{ lb./in.}^2$$

$$S_c = 636.6\left(\frac{2 \times 4.254}{5 \times 0.254} - 1\right) = 636.6 \times 5.699 = 3,628 \text{ lb./in.}^2$$

Problem

A hook of circular section is 3 inches in diameter. The inner radius of curvature is 4 inches and the distance from the center of the section to the load line is 3 inches. If the maximum allowable unit stress is 10,000 pounds per square inch, what is the safe load? *Ans.* 6,400 pounds.

193. Hook of Trapezoidal Section.—Hooks are frequently made of trapezoidal section with the larger base toward the center of curvature. In the actual hooks the corners are rounded as in

Fig. 218. Such a hook may be calculated as if it were the full trapezoidal section and the bending stress in the actual hook may then be computed by multiplying the stress obtained from the full trapezoid by the ratio of the moment of inertia of the full trapezoid to the moment of inertia of the actual section. This moment of inertia may be calculated with respect to the center of gravity or with respect to the neutral axis of the full section.

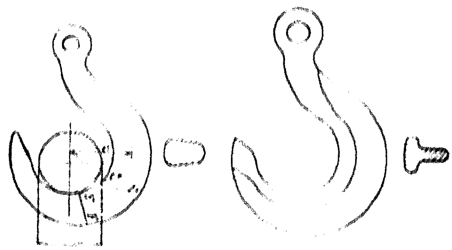


FIG. 218.—Hooks.

Example

A hook of trapezoidal section is 2 inches wide at the concave surface, 1 inch wide at the convex surface and 4 inches deep between these surfaces at the section most remote from the line of the load. The inner radius for this section is 4 inches and the line of the load is 2 inches from the concave surface. Find the unit stress in the extreme fibers when the load is 8,000 pounds.

To find the bending stress:

$$R_1 = 4 \text{ inches}, \ R_2 = 8 \text{ inches}, \ C = 12 \text{ inches}, \ m = -\frac{1}{4}.$$

$$R_0 = \frac{24}{4.3178} = 5.558 \text{ inches}.$$

$$S_1 = \frac{(4 - 5.558)M}{4(37.333 - 105.348 + 66.696)} = \frac{1.558\,M}{5.276}.$$

The center of gravity is $1\frac{7}{9}$ inches from the concave surface so that the moment arm is $3\frac{7}{9}$ inches.

$$S_1 = 0.2952 \times 8{,}000 \times 3\tfrac{7}{9} = 8{,}920 \text{ lb./in.}^2$$

$$\frac{P}{A} = \frac{8{,}000}{6} = 1{,}333.$$

$$S_t = 10{,}253 \text{ lb./in.}^2$$

Problem

Solve the example above for the unit stress in the extreme outer fibers.

194. Variation of Dimensions and Curvature of Hooks.—In a hook or curved rod subjected to tension or compression parallel

to its length, the sections most remote from the line of the load must have the greatest section. Since the unit stress at the concave surface is made smaller by increasing the radius of curvature it is desirable to make the radius of the more remote sections as great as practicable. In Fig. 219 the section at A has the inner radius OA while the section at B has the smaller radius CB, and the section at E, where the moment is small, has a still smaller radius.

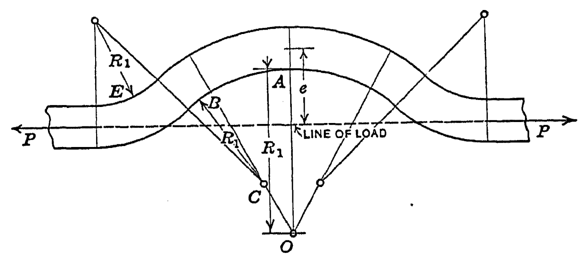


FIG. 219.—Curved bar.

The same principles may be applied to the design of a hook, but the increased length needed when the larger radius is used may require more material than would be necessary with the smaller radius and larger section.

In Fig. 218, I, the section CD is subjected to considerable shearing stress which must be taken into account in designing the section. The section EF is subjected, not only to shear, but also to a concentrated compressive stress for which due allowance must be made in the design.

CHAPTER XIX

CENTER OF GRAVITY

195. Center of Gravity.—When each of the particles which compose a body or system of bodies is subjected to a force which is proportional in magnitude to the mass of the particle and parallel to the similar forces in every other particle, the line of application of the resultant of these forces passes through the *center* of *gravity* of the body or system.

The location of the center of gravity is determined from the intersection of two such resultants.

Fig. 220 represents three particles of relative masses 2, 3 and 4, united by weightless rods to form a single body. In Fig. 220, I, these particles are subjected to forces directed vertically downward. The resultant of these forces is a force of 9 units along the line CD. The center of gravity is located at some point on this line. In Fig. 220, II, the forces are horizontal, and their resultant is a horizontal force of 9 units along the line EF. The point O at the intersection of EF with CD is the center of gravity.

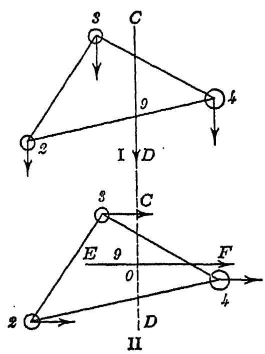


FIG. 220.—Center of gravity of three particles.

The center of gravity is also called the *center* of *mass*.

196. Determination of the Center of Gravity by Balancing.—The force with which the earth attracts the particles of a body is proportional* to the mass of each particle. These forces are directed toward the center of the earth, so that for bodies of ordinary dimensions they may be regarded as parallel, within the limits of accuracy of our measurement. The resultant force of gravity on any body passes through the center of gravity. A body may be held in equilibrium by a single force provided that

* There is a difference in the attraction of the earth due to difference in the distance of the various particles from the center of the earth amounting to about 1 part in 10 million for a difference of 1 foot. This is negligible for ordinary bodies. It would not be negligible in the case of a mountain.

force is along the line of the resultant of all the other forces. When a body is supported by a flexible cord or by a point about which it is free to turn without friction, the center of gravity must be on the vertical line through the point of application of the cord or point (provided, of course, no forces are acting except gravity and the cord or point).

Fig. 221 shows that same body as Fig. 220. In Fig. 221, I, it is supported at C by a cord. A plumb line let fall from C passes through the center of gravity. In Fig. 221, II, it is supported on a point or knife-edge at E, and turns under the action of gravity until its center of gravity comes directly below the point of support. The intersection of the plumb line from E with the line CD (previously marked in any convenient way) gives the center of gravity O.

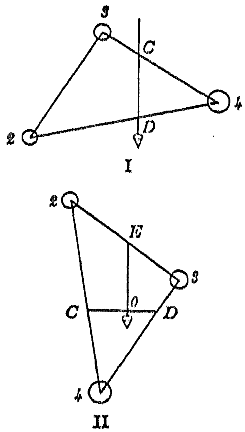


FIG. 221.—Location of center of gravity.

This method of finding the location of the center of gravity is of little practical use, owing to the fact that the point to be found is usually surrounded by solid material, making it necessary to find the intersection of three planes instead of the intersection of two lines. It is useful in relatively long bodies, especially if there are some plane surfaces to use as planes of reference. Fig. 222 represents a beam balanced on a knife-edge. The center of gravity is in the vertical plane of the knife-edge.

197. Center of Gravity by Moments.—In theoretical discussions the center of gravity is usually located by moments. The plane of application of the resultant of any set of forces may be determined by dividing the sum of the moments with respect to any axis by the resultant force.

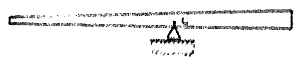


FIG. 222.—Center of gravity by balancing.

Example

Four masses are attached to a straight rod. These are: 16 pounds at the left end, 20 pounds at 7 feet, 8 pounds at 10 feet, and 12 pounds at 14 feet from the left end. The rod weighs 4 pounds and its center of gravity is 8 feet from the left end. Find the center of gravity of the rod and bodies combined.

Taking moments about an axis through the left end perpendicular to the length of the rod, and arranging the work in a convenient form:

	Mass	Moment arm	Moment
	16	0	0
	20	7	140
	4	8	32
	8	10	80
	12	14	168
Total	60 pounds.		420 foot-pounds.

Dividing 420 by 60 gives 7 feet as the distance of the center of gravity from the left end.

198. Center of Gravity of Bodies in a Straight Line.—The example of the preceding article is an illustration of the process of finding the center of gravity of a number of particles which lie on a straight line, or of a number of bodies, the center of gravity of each of which is in a straight line. If m_1, m_2, m_3 are the masses of the bodies, and x_1, x_2, x_3 are the distances of their respective centers of gravity from some point in their line,

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3},$$

where $\bar{x}$ is the distance of the center of gravity of the combination from the point taken as the origin.

Problems

1. The following masses are in a straight line: 8 pounds at 3 feet, 12 pounds at 6 feet, 9 pounds at 10 feet, 11 pounds at 12 feet. Using an axis through 0 feet as the origin, find the center of gravity. Check by using an axis through 5 feet. Arrange work as in the example of the preceding article. *Ans.* $\bar{x} = 7.95$ feet.

2. Find the center of gravity of 40 pounds at 5 feet, 48 pounds at 12 feet, and 65 pounds at 17 feet. Check the result.

A uniform rod or bar has its center of gravity at the middle of its length. In finding the moment of a uniform bar with respect to an axis which does not go through the end, consider the entire bar as a single body and do not divide it into two portions at the axis. For instance, it is required to find the moment of a uniform bar 12 feet long and weighing 8 pounds per foot with respect to an axis which cuts the bar 4 feet from the left end. The entire bar weighs 96 pounds and the moment arm is 2 feet, making the moment 192 foot-pounds. If the portions are taken separately,

the portion to the right of the origin of moments weighs 64 pounds and has a moment arm of 4 feet making a clockwise moment of 256 foot-pounds, while the portion to the left of the origin weighs 32 pounds and has a moment arm of 2 feet making a counter-clockwise moment of 64 foot-pounds. The total moment is $256 - 64 = 192$ foot-pounds clockwise. It is evident, therefore, that the first method, whereby the entire bar is considered as a unit, is preferable.

Problem

3. A uniform bar weighing 20 pounds per foot is 12 feet long and carries a mass of 60 pounds 3 feet from the left end and a mass of 150 pounds at the right end. Find the center of gravity of the combination, and check the result.

199. Center of Gravity of Bodies in a Plane.—When it is desired to find the center of gravity of a number of bodies which

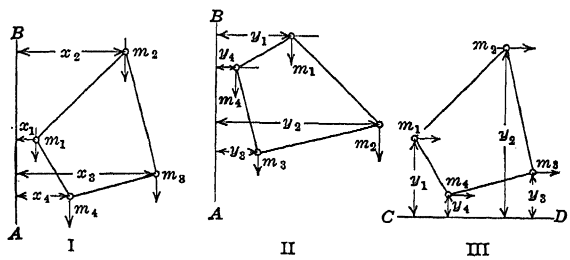


FIG. 223.—Center of gravity by moments.

lie in a plane, it is necessary to find the moments with respect to two axes. Fig. 223 represents four bodies of masses m_1, m_2, m_3, and m_4 in the plane of the paper. If a point in AB, Fig. 223, I, be taken as the origin and the forces of gravity on the masses have the direction of the arrows, the moment arms of these forces are x_1, x_2, x_3, and x_4, and the equation of the preceding article gives $\overline{x}$. To find the other coördinate of the center of gravity, the system may be rotated 90 degrees to the position of Fig. 223, II. Instead of rotating the system of bodies, it is better to regard the direction of the forces as rotated, as shown by Fig. 223, III. If then, any point in CD be taken as the origin, the moment arms are y_1, y_2, y_3, and y_4 and

$$\overline{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_3 + m_4}.$$

In each of these cases the axis of moments is regarded as being

perpendicular to the plane of the paper. Instead of this, the axes may be regarded as parallel to the plane of the paper and the forces normal thereto. In Fig. 223, I, the line AB in the plane of the paper may be taken as the axis of moments. The lengths of the moment arms are not changed when this is done. In fact, to get $\bar{x}$, any axis may be taken provided it lies in a plane perpendicular to x_1, x_2, etc. In finding the center of gravity, it is customary to speak of moments with respect to a plane. To find one coördinate of a center of gravity, multiply each mass by the distance of its center of gravity from some plane of reference. The sum of these moments divided by the sum of the masses gives the required distance.

Problems

1. A body is composed of three particles in the same plane

Mass	x	y
3	5	7
2	4	8
5	3	6

Find x and y.

$$\bar{x} = \frac{3 \times 5 + 2 \times 4 + 5 \times 3}{3 + 2 + 5} = \frac{38}{10} = 3.8,$$

$$\begin{aligned} 3 \times 7 &= 21 \\ 2 \times 8 &= 16 \\ 5 \times 6 &= 30 \\ \hline 10\,\bar{y} &= 67 \\ \bar{y} &= 6.7 \end{aligned}$$

The second form of solution is preferable, especially when the numbers are large. It is still better to arrange the data and results in a single table omitting the multiplication and equality signs.

2. Find the coördinates of the center of gravity of

Mass	x	y	mx	my
12	4	3		
10	−3	4		
8	3	7		

$\bar{x} = 1.4$

$\bar{y} = ?$

3. Solve Problem 2 taking moments with respect to the planes $x = 2$, and $y = 3$.

200. Center of Gravity of a Plane Area.—In Strength of Materials the so-called center of gravity of a plane area is an important

factor. A plane area may be regarded as a plate of uniform thickness and density. In calculating the moment of such an area it is customary to regard the mass per unit area as unity.

The center of gravity of a few plane areas may be determined geometrically. If the area has a line of symmetry, the center of gravity lies in this line. If it has two lines of symmetry the center of gravity is at their intersection. The center of gravity of a circle is at the center. The center of gravity of the rectangle of Fig. 224 is at the intersection of the line of symmetry AB midway between the ends, with the line of symmetry CD midway between the sides.

A triangle, Fig. 225, may be regarded as made up of an indefinite number of infinitesimal rectangles such as AB. The center of gravity of each rectangle is at the middle of its length so that it is evident that the center of gravity of a triangle lies in the

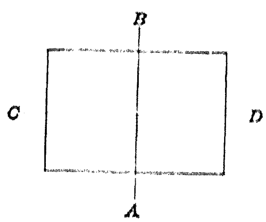


FIG. 224.—Center of gravity of rectangle.

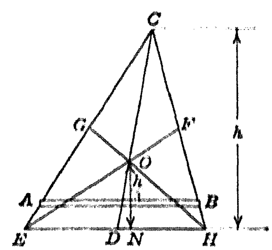


FIG. 225.—Center of gravity of triangle.

median line CD. In like manner it may be shown that the center of gravity lies in one of the other medians EF or GH. It must, therefore, lie at the intersection of the medians. It was proven in Geometry that two medians are trisected at their intersection. The line OG is one-third of GH. If h is the altitude of the triangle measured perpendicular to the base, the perpendicular distance ON, from the center of gravity to the base, is $\frac{h}{3}$.

The center of gravity of a parallelogram lies at the intersection of two lines parallel to the sides and midway between them.

Many important figures are made up of combinations of triangles, parallelograms, and circles.

Example

Find the distance of the center of gravity of a 6-inch by 5-inch by 1-inch angle section from the back of the legs.

The section may be divided into a 4-inch by 1-inch rectangle, and a 6-inch by 1-inch rectangle (Fig. 226). Taking moments with respect to the axis YY at the back of the 5-inch leg,

$$\begin{array}{rcrcl} 4 & \times & 0.5 & = & 2 \\ 6 & \times & 3.0 & = & 18 \\ \hline 10x & & & = & 20 \\ x & & & = & 2 \text{ inches} \end{array}$$

Problems

1. In the example above find y. *Ans.* $\bar{y} = 1.5$ inches.

2. Solve Problem 1 by dividing the section into two 5-inch by 1-inch rectangles.

3. Find the distance from the lower base of the center of gravity of the trapezoid, lower base 10 inches, upper base 6 inches, height 12 inches.

4. Solve Problem 3 by dividing the trapezoid into two triangles as shown in Fig. 227, II.

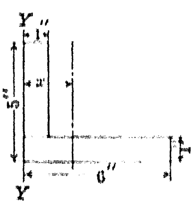


FIG. 226.—Center of gravity of angle section.

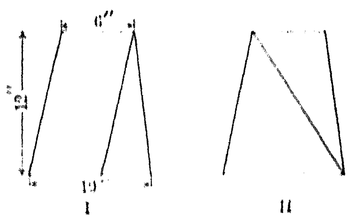


FIG. 227.—Center of gravity of trapezoid.

It is sometimes convenient to consider a given area as the difference between two areas. The angle section of Fig. 226 may be regarded as a 6-inch by 5-inch rectangle from which a 5-inch by 4-inch rectangle has been subtracted.

$$\begin{array}{rcr} \text{Area} & \text{Arm} & \text{Moment} \\ 30 & 3 & 90 \\ -20 & 3.5 & -70 \\ \hline 10x & = & 20 \\ x & = & 2 \end{array}$$

Problems

5. Solve Problem 3 by regarding the trapezoid as the difference between a 10-inch by 12-inch parallelogram and a triangle with a 4-inch base.

6. A rectangular board 20 inches long and 18 inches wide has a circular hole, 8 inches in diameter, cut out. The center of the hole is 5 inches from

the left end and 6 inches from the lower edge. Find the center of gravity of the remainder. Check by means of axes through the center of the hole.

$\bar{x} = 10.81$ inches from the left end.

7. A 12-inch circular board has an 8-inch hole with its center 1 inch from the center of the board. Find the distance of the center of gravity from the center of the board. Do not multiply by π to find the actual area of either circle.

8. Fig. 228 represents a standard 10-inch 15-pound channel section. Find the distance of the center of gravity of the section from the back of the web, and compare with handbook.

9. Look up dimensions of a 12-inch 30-pound channel and calculate the area and the location of the center of gravity.

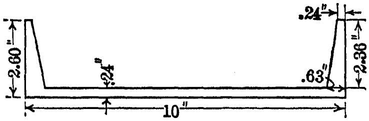


FIG. 228.—A channel section.

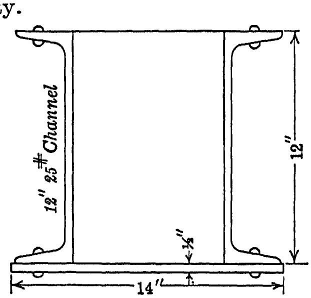


FIG. 229.—Plate and channel section.

10. A section is made of two 12-inch 25-pound channels and one 14-inch by ½-inch plate. Look up area of channels in handbook and compute the distance of the center of gravity of the section from the common surface of the plate and channels.

201. Center of Gravity of Area by Integration.—In the preceding article, centers of gravity have been found by dividing the area into a finite number of parts of each of which the area and location of its center of gravity is known. When the boundary of the area is such that it cannot be divided accurately into parallelograms, triangles, or circles, it is necessary to integrate to obtain exact results. Fig. 230 represents several areas. If x' be the distance from the plane with respect to which moments are taken to the *center of gravity* of an element of area dA, the moment of that element is $x'dA$ and the moment of the entire area is $\int x'dA$. The total area is $\int dA$, so that

$$\bar{x} = \frac{\int x'dA}{\int dA} = \frac{\int x'dA}{A}. \qquad (1)$$

$$\bar{y} = \frac{\int y'dA}{A}. \qquad (2)$$

In Fig. 230, I, x' is the same as the x of the curve at the intersection of the element of area; but y' is one-half of the y of this point. (If the element of area be regarded as of considerable width Δx, then $x' = x + \frac{\Delta x}{2}$, but the limiting value of this, as Δx approaches 0 as a limit, becomes x. See *Proceedings of the Society for the Promotion of Engineering Education*, June, 1916, for another explanation of these principles.)

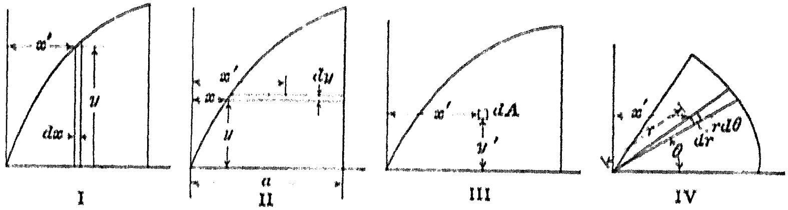


FIG. 230.—Elements of area.

In Fig. 230, II, the element of area is horizontal and y' is equal to y. On the other hand, $x' = x + \frac{a - x}{2} = \frac{a + x}{2}$. In Fig. 230, III, which is intended for double integration, $x' = x$ and $y' = y$.

In Fig. 230, IV, the element of area is $rd\theta dr$, $x' = r \cos \theta$, and $y' = r \sin \theta$.

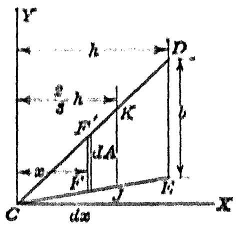


FIG. 231.—Center of gravity of triangle.

Problem

1. Find the center of gravity of a triangular area, of base b and altitude h by integration. We will take the origin at the vertex C and so place the base ED of length b that it shall be parallel to the Y axis. The element of area is a strip of width dx. From similar triangles, the length of this strip FF' is $\frac{bx}{h}$.

$$dA = \frac{bx}{h}dx;$$

$$\bar{x} = \frac{\frac{b}{h}\int x^2dx}{\frac{b}{h}\int xdx} = \frac{\frac{b}{h}\left[\frac{x^3}{3}\right]_0^h}{\frac{b}{h}\left[\frac{x^2}{2}\right]_0^h} = \frac{\frac{bh^2}{3}}{\frac{bh}{2}}$$

We recognize the denominator of the last term of (3) as the area of the triangle, which shows that our increment of area was taken correctly and the proper limits were used.

2. Find the distance from the Y axis of the center of gravity of the plane area bounded by the X axis, the parabola $y^2 = 4x$, and the ordinate $x = 9$ (Fig. 232, I.).

The element of area is ydx, and the moment arm is x of the curve

$$\bar{x} = \frac{\int xydx}{\int ydx}. \tag{1}$$

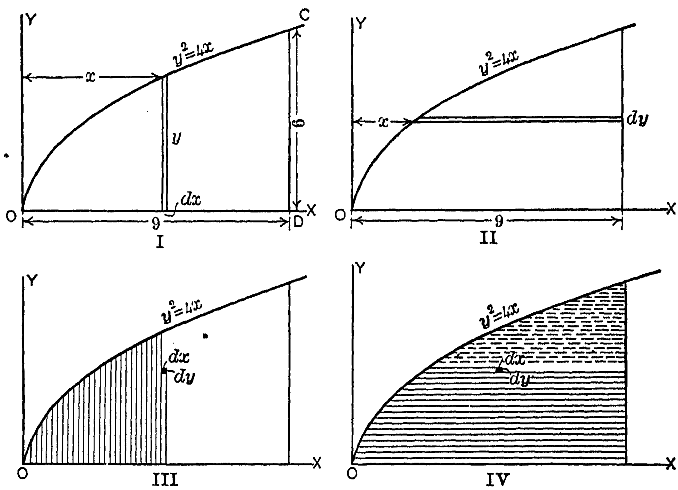


FIG. 232.—Center of gravity of area bounded by a parabola.

Since there are two variables in the integrals of (1), we must eliminate one of these by substituting its value in terms of the other from the equation of the curve. Solve first by eliminating y and integrating between the proper limits for x; then solve by eliminating x and dx and integrating between the proper limits for y. Compare results. The result should be greater than one-half of 9 and should be less than 6. Why?

3. Solve Problem 2 for $\bar{y}$, using the horizontal element of area of Fig. 232, II. The result should be greater than 2 and less than 3. Why?

4. Solve Problem 3 for $\bar{y}$, using the vertical element of area of Fig. 232, I.

5. Solve Problem 2 for $\bar{x}$, using the horizontal element of area of Fig. 232, II.

6. Solve Problem 2 for $\bar{x}$ by double integration, integrating first with respect to y. After putting in the limits for y compare the result of this first integration with the first step of Problem 2 (Fig. 232, III).

7. Solve Problem 2 by double integration, integrating x first. Compare with Problem 5 (Fig. 232, IV).

TABLE XXIII

Table XXIII gives the area, the location of the center of gravity, and the moment with respect to a plane through the left end, for some plane figures. The first two of these are of especial importance in finding the deflection of a beam by the method of Area Moments, and the others are sometimes used for this purpose.

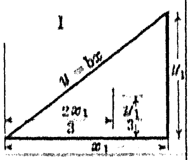
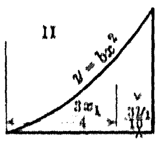
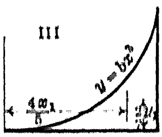
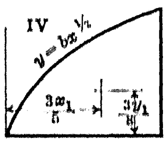
	EQUATION	AREA	M_x	$\bar{x}$	$\bar{y}$
	$y = bx$	$\frac{x_1 y_1}{2}$	$\frac{bx_1^3}{3} = \frac{x_1^2 y_1}{3}$	$\frac{2x_1}{3}$	$\frac{y_1}{3}$
	$y = bx^2$	$\frac{x_1 y_1}{3}$	$\frac{bx_1^4}{4} = \frac{x_1^2 y_1}{4}$	$\frac{3x_1}{4}$	$\frac{3y_1}{10}$
	$y = bx^3$	$\frac{x_1 y_1}{4}$	$\frac{bx_1^5}{5} = \frac{x_1^2 y_1}{5}$	$\frac{4x_1}{5}$	$\frac{4y_1}{14}$
	$y = bx^{\frac{1}{2}}$	$\frac{2x_1 y_1}{3}$	$\frac{2bx_1^{\frac{5}{2}}}{5} = \frac{2x_1^2 y_1}{5}$	$\frac{3x_1}{5}$	$\frac{3y_1}{8}$

FIG. 233.—Center of gravity of some plane figures.

Problems

8. Verify the values of Table XXIII.

9. Find the moment with respect to the Y axis of the area enclosed by the X axis, the line $y = bx$, and the ordinate $x = c$ and $x = d$. Solve by means of the Table XXIII and check by dividing the area into a triangle and a rectangle.

$$Ans.\ M_x = \frac{b(d^3 - c^3)}{3}.$$

10. Find $\bar{x}$ of the area bounded by the Y axis, the line $y = 6$, the hyperbola $xy = 12$, the line $x = 12$, and the X axis, using a vertical strip as the increment of area.

$$Ans.\ \bar{x} = \frac{12 + \int xy\,dx}{12 + \int y\,dx} = \frac{12 + 12\int dx}{12 + \int \frac{12}{x}dx} = \frac{12 + 12[x]_2^{12}}{12 + 12[\log x]_2^{12}} = 3.94.$$

11. Find $\bar{x}$ of a 60-degree sector of a circle of radius a with the X axis as one of the bounding lines (Fig. 234). Solve by polar coördinates, integrating first with respect to r. (The order of integration is immaterial in this case, as the limits of one variable are independent of the other variable. Where this is not the case, integrate first with respect to r.)

Ans. $\bar{x} = \dfrac{\sqrt{3}\,a}{\pi} = 0.551\,a.$

12. Using the value of $\bar{x}$ from Problem 11, find $\bar{y}$ without integrating.

Ans. $\bar{y} = \dfrac{a}{\pi}.$

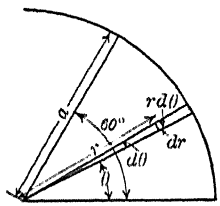


FIG. 234.—Center of gravity of a sector of a circle.

13. Solve Problem 11 for $\bar{x}$ if the sector is so placed that the X axis bisects it. Compare with results of the preceding problems.

14. Find the center of gravity of segment of a circle of radius 10 bounded on one side by a straight line at a distance 5 from the center of the circle. Solve by rectangular coördinates, using strips parallel to the boundary line as increments of area. *Ans.* $\bar{x} = 7.05.$

Using only the half above the X axis and calling the radius a:

$$x = \frac{\int xy\,dx}{\int y\,dx} = \frac{\int (a^2 - x^2)^{\frac{1}{2}} x\,dx}{-a^2 \int \sin^2 \theta\,d\theta}$$

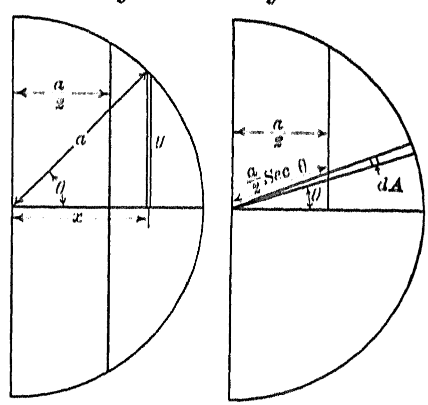


FIG. 235.—Center of gravity of a segment of a circle.

$$\bar{x} = \frac{-\dfrac{\left[\left(a^2 - x^2\right)^{\frac{3}{2}}\right]_{\frac{a}{2}}^{a}}{3}}{-a^2\left[\dfrac{\theta}{2} - \dfrac{\sin 2\theta}{4}\right]_{\frac{\pi}{3}}^{0}} = \frac{\dfrac{a^3\sqrt{3}}{8}}{a^2\left[\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{8}\right]}.$$

The independent variable is changed in the denominator and might also be changed in the numerator. Why is the upper limit in the denominator 0 and not $\frac{\pi}{3}$? Explain the geometric meaning of each term in the denominator.

15. Solve Problem 14, by double integration with polar coördinates (Fig. 235, II).

16. Find the center of gravity of a semicircular area of radius a.

Ans. $\bar{x} = \dfrac{4a}{3\pi} = 0.4244\,a.$

202. Center of Gravity by Experiment.—Sections of materials subjected to bending are frequently of such form that it is difficult to express the boundaries in mathematical symbols. In that case, the center of gravity may be found by cutting the section out of cardboard or sheet metal and balancing on a knife-edge. Where considerable accuracy is desired, especially when the section is small, it is best to cut out a short portion of the beam between two transverse planes and locate its center of gravity by balancing on the beam of a delicate balance.

Fig. 236, I, represents a body on a beam balanced by the poise

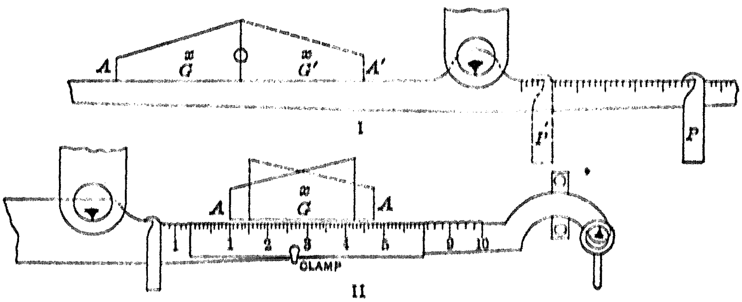


FIG. 236.—Center of gravity by balancing.

in the position shown. If the body is then turned end for end on the beam with the edge O not changed, so as to come into the position shown by the dotted lines, the center of gravity is moved a distance GG', which is twice its distance from O. To get a balance, the poise P must be moved to the dotted position P'. If the mass of the body is known in terms of the poise, the distance GG' may easily be calculated. Instead of rotating about the end O, any vertical line may be used as the line of reference whose position on the beam is not changed.

Problem

1. A body weighing 4.50 pounds is balanced on a scale beam. When turned about a vertical line through the end nearest the knife-edge, the ap-

parent change in weight is 0.576 pound. The distance from the central knife-edge to the end knife-edges is 10 inches? How far is the center of gravity from the line about which it turned? *Ans.* 0.64 inch.

The method just given requires that we know the weight of the poise and the value in inches of a division on the scale, or that we know the distance from the central knife-edge to the knife-edge upon which 1 pound weighs 1 pound. Another method is that shown by Fig. 236, II. The body is placed on the beam as before and moved till equilibrium is secured with some convenient weight on the opposite end. It is then turned end for end and moved along the scale beam until the same balance is secured, with all other weights unchanged. The center of gravity is now in the same position which it occupied before turning. If the position of any point such as A is noted before turning and again after turning, the difference of these two positions is twice the horizontal distance of A from the center of gravity. This may be done with great accuracy on the beam of a platform scale by clamping to the beam a small steel scale for determination of the displacement of the points as shown in the figure.

Problem

2. A body is balanced on a scale beam. When turned around and again balanced, it is found that the point originally at the left end is displaced 3.32 inches. How far is the center of gravity from this end?

Ans. 1.66 inches.

CHAPTER XX

MOMENT OF INERTIA

203. Definition.—The moment of inertia of a body with respect to an axis is the sum of the products obtained by multiplying the mass of each particle of the body by the square of its distance from the axis. If m is the mass of any particle, and r is its distance from the axis,

$$I = \Sigma mr^2.$$

For a continuous body, the definition expressed mathematically is

$$I = \int r^2 dM, \qquad \text{Formula XXXIII}$$

where I is the moment of inertia and dM is any element of mass

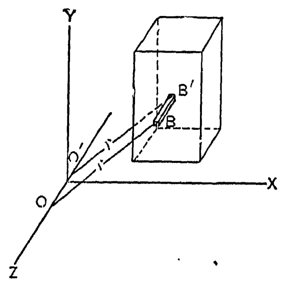


FIG. 237.—Element of volume.

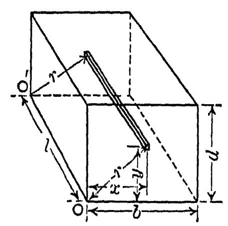


FIG. 238.—Moment of inertia of parallelopiped.

(finite or infinitesimal), all parts of which are at a distance r from the axis.

In Fig. 237, the Z axis is taken as the axis of inertia for the solid. The element BB' extending entirely through the body parallel to the Z axis is the element of mass of Formula XXXIII. The element of mass might have the form of a hollow cylinder of radius r, and thickness dr. It could, of course, be of infinitesimal dimensions in three directions, in which case the volume would be represented by $dx\, dy\, dz$ in *rectangular coördinates*, by $r\, d\theta\, dr\, dz$ in *cylindrical* or *mixed coördinates*, and by $r^2 \sin\, \theta\, d\theta\, d\phi\, dr$ in spherical coördinates (with the Z axis as the axis of the sphere from which θ is measured). When taking moment of inertia

with respect to the Z axis the values of r^2 of Formula XXXIII in terms of the coördinates of the element are:

$$x^2 + y^2 \text{ for rectangular coördinates,}$$
$$r^2 \quad \text{for cylindrical coördinates,}$$
$$r^2 \sin^2 \theta \text{ for spherical coördinates.}$$

The element BB' of Fig. 237 may be regarded as an example of rectangular coördinates after integration with respect to Z. If its cross-section were of the form of an element of area in polar coördinates it would be an example of the cylindrical element of volume after one integration. A second integration of this element of volume with respect to θ would give the hollow cylinder of thickness dr.

Problems

1. Find the moment of inertia of a rectangular parallelopiped of width b, height d, and length l, with respect to an edge parallel to its length (Fig. 238) by double integration. *Ans.* $I = \frac{\rho bdl}{3}(b^2 + d^2) = \frac{M}{3}(b^2 + d^2)$, where ρ is the mass per unit volume and M is the total mass.

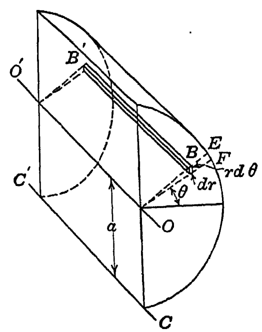


FIG. 239.—Moment of inertia of cylinder.

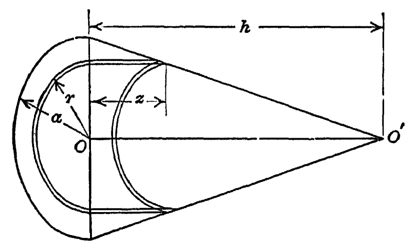


FIG. 240.—Moment of inertia of a cone.

2. Find I of a homogeneous solid cylinder of length l and radius a with respect to the axis of revolution (OO', Fig. 239). *Ans.* $I = \frac{\pi\rho l a^4}{2} = \frac{Ma^2}{2}$.

This is a case of cylindrical cöordinates. The element of volume for double integration has a length l and a cross-section $r\,d\theta\,dr$. Integrating first with respect to r gives a wedge-shaped element between the planes whose traces on the front are the dotted lines OE and OF. The second integration builds up the cylinder of a series of such wedges.

If θ be integrated first between the limits 0 and 2π, we get a hollow cylinder of radius r and thickness dr. The volume of this hollow cylinder is $2\pi rl\,dr$, and its moment of inertia with respect to the axis OO' is $2\pi\rho l r^3\,dr$, which might have been obtained directly without integrating.

3. Find the moment of inertia of a right cone of height h and radius of base a with respect to the axis of revolution, by a single integration using a hollow cylinder as the element of volume.

$$Ans.\ I = 2\pi\rho r^3 z\,dr = 2\pi\rho h\int_0^a\left(r^3 - \frac{r^4}{a}\right)dr = \frac{\pi\rho a^4 h}{10} = \frac{3Ma^2}{10}.$$

4. Find the moment of inertia of a homogeneous solid sphere of radius a with respect to a diameter by double integration. Use as the element of volume a ring of radius r and cross-section $dr\,dz$ and integrate first with respect to z. *Ans.* $I = \frac{2}{5}Ma^2$.

5. Solve Problem 4, integrating first with respect to r. Show that this is the same as a single integration using a disk or short cylinder of length dx as the element of volume, and applying the results of Problem 2.

6. Solve Problem 3 by a single integration building the cone of flat disks parallel to the base, the moment of inertia of each disk being from Problem 2, $\frac{\pi\rho r^4}{2}dx$, where r is measured from the axis to the surface.

204. Radius of Gyration.—The radius of gyration may be defined algebraically by the equations:

$$Mk^2 = I, \tag{1}$$

$$k^2 = \frac{I}{M}, \tag{2}$$

where k is the radius of gyration.

The radius of gyration is the distance from the axis at which the entire mass could be concentrated and leave the moment of inertia unchanged. In the case of a homogeneous solid cylinder with respect to the axis of revolution,

$$I = \frac{Ma^2}{2};$$

$$k^2 = \frac{a^2}{2};$$

$$k = \frac{a}{\sqrt{2}} = 0.7071\,a.$$

If the entire mass of a solid cylinder of radius a were condensed into a hollow cylinder of radius 0.707 a and negligible thickness, or into a single filament at a distance of 0.707 a from the axis, the moment of inertia in each case would be the same as that of the solid cylinder.

Problems

1. Find the square of the radius of gyration of a homogeneous solid cylinder of 10 inches radius. *Ans.* $k^2 = 50$.

2. Find the radius of gyration of a parallelopiped of breadth 8 inches and depth 10 inches with respect to an axis parallel to the length along one edge. *Ans.* $k = 7.39$ inches.

3. A solid cylinder of 8 inches radius and weighing 60 pounds is coaxial with a second solid cylinder of 10 inches radius and weighing 40 pounds. Find the radius of gyration of the two cylinders with respect to their common axis. *Ans.* $k = 6.20$ inches.

4. Find the radius of gyration of a homogeneous hollow cylinder of outside radius a and inside radius b with respect to the axis of revolution.

$$Ans.\ k = \sqrt{\frac{a^2 + b^2}{2}}.$$

5. By integration, find the moment of inertia of a homogeneous solid cylinder with respect to an element of the curved surface as an axis (CC'', Fig. 239). *Ans.* $I = \frac{3}{2} Ma^2$.

205. Transfer of Axis.—When it is necessary to find the moment of inertia with respect to some axis for which the equation of the solid is complicated, the integration becomes laborious. Usually it is best to first find the moment of inertia with respect

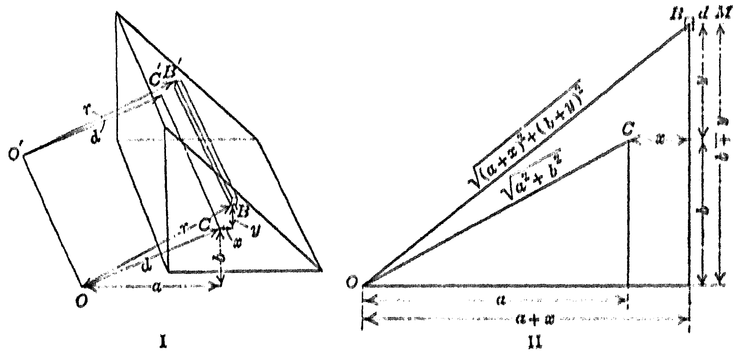


FIG. 241.—Transfer of axis.

to an axis giving the simplest expression for the equation of the solid and then transfer to the new axis. If CC' is an axis passing through the center of gravity of a solid and OO' is a *parallel axis* at a distance d from it, we will prove that

$$I = I_0 + Md^2, \qquad \text{Formula XXXIV}$$

where I is the moment of inertia with respect to OO' and I_0 is the moment of inertia with respect to CC'.

Let BB' (Fig. 241) be an element of mass parallel to the axes. Its coördinates with respect to the axis CC' are (x, y). Let the coördinates of the center of gravity with respect to the axis OO' be (a, b) so that $d = \sqrt{a^2 + b^2}$. With respect to OO' the value of r^2 in the expression

$$I = \int r^2 dM$$

is

$$r^2 = (a + x)^2 + (b + y)^2.$$

With respect to OO' the expression for the moment of inertia is

$$I = \int (a^2 + 2\,ax + x^2 + b^2 + 2\,by + y^2)\,dM. \qquad (1)$$

$$I = \int (x^2 + y^2)dM + (a^2 + b^2)\int dM + 2\,a\int xdM + 2\,b\int ydM \quad (2)$$

We recognize the first term of the second member of (2) as the moment of inertia with respect to CC'. The second term is $(a^2 + b^2)\,M$, which is Md^2.

The third term, $2\,a\int xdM$, is zero; xdM being the moment of dM with respect to a vertical plane through CC', and the sum of these moments is zero when the center of gravity falls in this vertical plane.

Also

$$\bar{y} = \int \frac{ydM}{M}.$$

When y is measured from the center of gravity $\bar{y} = 0$ and $\int \frac{ydM}{M} = 0$, consequently the last term of (2) is zero and equation (2) becomes Formula XXXIV.

Problems

1. Find the moment of inertia of a homogeneous solid cylinder of mass M and radius a with respect to an axis in its surface parallel to the axis of the cylinder. *Ans.* $I = \frac{3\,Ma^2}{2}$.

2. Find the moment of inertia of a homogeneous solid cylinder 8 inches in diameter with respect to an axis parallel to the axis of the cylinder and 10 inches therefrom. *Ans.* $I = 1{,}728\,\pi\rho l = 5{,}429\rho l$.

206. Moment of Inertia of a Plane Area.—The moment of inertia of a plane area may be defined mathematically by the expression

$$I = \int r^2 dA. \qquad (1)$$

It is equivalent to the moment of inertia of a thin plate of mass unity per unit area and of such small thickness that the square of the thickness is negligible compared with the square of the other dimensions.

There are two important cases of the moment of inertia of a plane area; in the first case the axis lies in the plane of the area; in the second case the axis is normal to the plane.

The moment of inertia of a plane area with respect to an axis in

its plane is an important constant in all problems concerning the strength or deflection of beams or columns. This moment of inertia is represented by the letter I.

The moment of inertia of a plane area with respect to an axis perpendicular to its plane is called the *polar moment of* inertia. The polar moment of inertia is an important factor in all problems concerning the strength of shafting in torsion and the amount of twist of such shafts. It is represented by the letter J.

The polar moment of inertia of a plane area is equivalent to the moment of inertia of a solid plate of the same dimensions and of such thickness that the product of the thickness and density is unity. The radius of gyration of a plane area is given by:

$$k^2 = \frac{I}{A}, \tag{2}$$

$$k^2 = \frac{J}{A}, \tag{3}$$

which are the same as in the case of solids with the area used instead of the mass.

Formula XXXIV for the transfer of axes is modified in the same way,

$$I = I_0 + Ad^2. \tag{4}$$

Problems

1. By integration find the moment of inertia of a rectangle of breadth b and depth d with respect to the side b. *Ans.* $I = \dfrac{bd^3}{3}$.

2. By transfer of axis find the moment of inertia of a rectangle of sides b and d with respect to an axis in the plane of the area parallel to b and passing through the center of the rectangle. *Ans.* $I = \dfrac{bd^3}{12}$.

3. Find the moment of inertia and radius of gyration of a 6-inch by 10-inch rectangle with respect to an axis 2 inches outside the rectangle and parallel to a 6-inch edge. Solve by means of the answer to Problem 2 and transfer of axis. Check by the answer to Problem 1, subtracting the moment of inertia of a 6-inch by 2-inch rectangle from that of a 6-inch by 12-inch rectangle. *Ans.* $I = 3{,}440$ inches.^{4}

4. By integration with polar coördinates find the expression for the moment of inertia of a circular area of radius a with respect to a diameter.

Ans. $I = \dfrac{\pi a^4}{4}$.

The answers to Problems 1, 2, and 4 should be memorized.

5. Find the radius of gyration of a circular area of radius a with respect to a diameter. *Ans.* $\dfrac{a}{2}$.

6. Find the polar moment of inertia of a circle of radius a with respect to an axis through its center. Show that the radius of gyration is the same as that of a solid cylinder of the same radius. Find the relation between the polar moment of inertia and the moment of inertia of the same circle with respect to a diameter. *Ans.* $J = 2I$.

7. Find the moment of inertia of a triangle of base b and altitude h with respect to an axis through the vertex parallel to the base. Solve by integration. *Ans.* $I = \frac{bh^3}{4}$.

8. By transfer of axis find the moment of inertia of the triangle of Problem 7 with respect to an axis through the center of gravity parallel to the base. *Ans.* $I_0 = \frac{bh^3}{36}$.

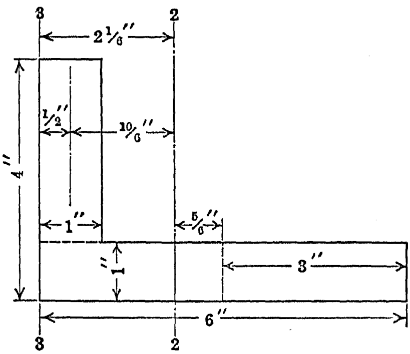


FIG. 242.

9. By transfer of axis find the moment of inertia of a triangle of base b and altitude h with respect to the base. *Ans.* $I = \frac{bh^3}{12}$.

10. Find the moment of inertia of the trapezoid of lower base 16 inches, upper base 10 inches and height 6 inches with respect to the lower base. Solve by dividing the trapezoid into a parallelogram and a triangle and use the answers to Problems 1 and 9.

11. Find the moment of inertia of the trapezoid of Problem 10 with respect to an axis through the center of gravity parallel to the base.

12. Find the moment of inertia of a 6-inch by 4-inch by 1-inch angle section with respect to an axis through the center of gravity parallel to the 4-inch leg.

Divide the section into two rectangles. Find the moment of inertia of each rectangle with respect to a line through *its* center of gravity parallel to the 4-inch leg and transfer to the required axis. Dividing the section into a 6-inch by 1-inch and a 1-inch by 3-inch rectangle, Fig. 242,

$$\frac{1 \times 6^3}{12} + 6\left(\frac{5}{6}\right)^2 = 22\frac{1}{6},$$

$$\frac{3 \times 1^3}{12} + 3\left(\frac{10}{6}\right)^2 = 8\frac{7}{12},$$

$$I_0 \text{ for axis 2–2} = 30\frac{3}{4} = 30.75 \text{ inches.}^4$$

The problem may also be solved by finding the moment of both rectangles with respect to one axis and the transferring to the center of gravity of the entire section. The axis 3–3 at the back of the 4-inch leg is a common base for both rectangles.

$$I = \frac{3 \times 1^3}{3} + \frac{1 \times 6^3}{3} = 73,$$

$$I_0 = 73 - 9\left(\frac{13}{6}\right)^2 = 73 - 42.25 = 30.75 \text{ inches.}^4$$

13. Solve Problem 12 by the last method, first finding the moment of inertia with respect to the right edge of the 4-inch leg as an axis and then transferring to the center of gravity.

14. Solve Problem 12 for the moment of inertia with respect to an axis through the center of gravity parallel to the 6-inch leg. Compare the results with the handbook.

15. A "plate-and-angle column" (see handbook) is made of four 4-inch by 3-inch by ½-inch angles and one 12-inch by ½-inch plate. The angles are riveted to the plate, the back of the longer legs being ⅛-inch above and below the edges of the plate. Taking the moments of inertia and location of centers of gravity from handbook tables of angle sections, find the moment of inertia of the entire section with respect to the two lines of symmetry.

16. Look up formula for the moment of inertia of standard channel section with respect to axis through the center of gravity perpendicular to the web. Derive this formula by means of the formulas for rectangles and triangles. The slope of the flange of a standard channel section is one-sixth.

17. From the dimensions given in the handbook find the moment of inertia of a 15-inch 33-pound channel for axis through the center of gravity perpendicular to the web.

18. A "plate-and-channel column" is made of two 15-inch 40-pound channels placed 12½ inches back to back (with toes out), and two 20-inch by ¾-inch plates. Taking the moments of inertia of the channels from the table of the properties of channel sections, find the moment of inertia of the section with respect to an axis parallel to the channel webs and midway between them and also with respect to an axis parallel to the plates through the centers of the channels.

19. Find the moment of inertia of a 20-inch 65-pound standard I-beam with respect to the two axes of symmetry. Derive the formulas used.

207. Product of Inertia.—The expression $\int xy\, dA$ is called the *product of inertia* of the area. It is represented algebraically by the letter H.

If an area is symmetrical with respect to either one of a pair of rectangular axes, its product of inertia with respect to that pair of axes is zero. Fig. 243 represents an area symmetrical with respect to the Y axis. If we integrate first with respect to x,

$$H = \tfrac{1}{2}\int\Big[x^2\Big]_{x_1}^{x_2} y\, dy = \tfrac{1}{2}\int [x_2^2 - x_1^2] y\, dy.$$

If the area is symmetrical with respect to the Y axis, the lower limit x_1 is numerically equal and opposite in sign to the upper limit x_2, and the squares are the same in magnitude and sign; consequently the term in the brackets vanishes and

$$H = 0.$$

When the product of inertia is known with respect to a pair of rectangular axes through the center of gravity of an area, it may be calculated for a second pair of parallel axes in the plane

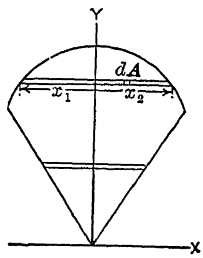


FIG. 243.—Symmetrical section.

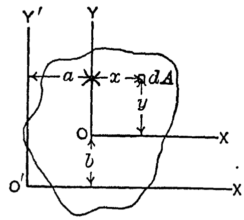


FIG. 244.—Transfer of axes for product of inertia.

of the area by a formula similar to XXXIV for the transfer of moments of inertia.

Let OX, OY, Fig. 244, be the original pair of axes through the center of gravity, and let (x, y) be the coördinates of an element dA with reference to these axes. Let $O'X'$, $O'Y'$ be a new pair of parallel axes. Let (a, b) be the coördinates of the center of gravity of the area with respect to the new axes.

If H is the product of inertia with respect to the new axes,

$$H = \int (a + x)(b + y)dA. \qquad (1)$$

$$H = ab\int dA + b\int x\,dA + a\int y\,dA + \int xy\,dA. \qquad (2)$$

$$H = abA + 0 + 0 + H_0. \qquad (3)$$

where H_0 is the product of inertia with respect to the axes through the center of gravity. Equation (3) is easily remembered from Formula XXXIV, replacing the square by the product.

If the center of gravity falls in the first or third quadrant with respect to the axes for which the product of inertia is desired, the product abA is positive, and H will be positive unless H_0 is negative and numerically greater than abA. If the center of gravity falls in the second quadrant abA is negative since a is negative; if it falls in the fourth quadrant abA is negative because b is negative.

Problems

1. By integration find the product of inertia of a rectangle of base b and altitude d with respect to the lower and left edges as axes. Check by equation (3). *Ans.* $H = \frac{b^2d^2}{4}$.

2. Solve Problem 1 for the lower and right edges as axes.

Ans. $H = -\frac{b^2d^2}{4}$.

3. By integration find the product of inertia of a 6-inch by 8-inch right-angled triangle with respect to the edges as axes (Fig. 245).

Ans. $H = 96$ inches.[4]

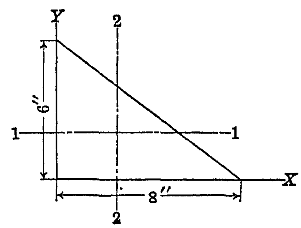


FIG. 245.—Product of inertia of a right triangle.

4. By transfer of axes find the product of inertia of the triangle of Problem 3 with respect to the axes 1–1, 2–2 through the center of gravity.

Ans. $H_0 = 96 - 128 = -32$ inches.[4]

5. Find the product of inertia of a 6-inch by 5-inch by 1-inch angle section with respect to axes through the center of gravity parallel to the legs. The section may be divided into two rectangles. The product of inertia of each of these with respect to the axes through their center of gravity is zero. Transferring to the axes 1–1, 2–2 and adding

$$H_0 = 4 \times (-1.5) \times 1.5 + 0 + 6 \times 1 \times (-1) + 0 = -9 - 6 = -15 \text{ inches.}^4$$

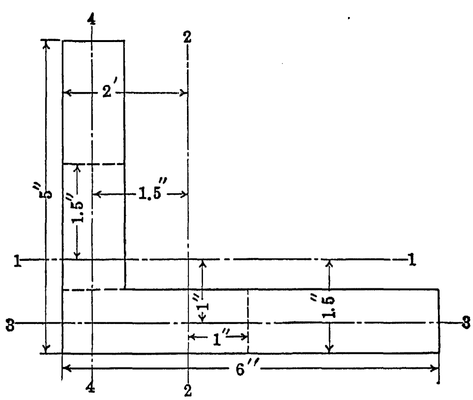


FIG. 246.

The problem may be solved more readily in another way. For the axes 3–3, 4–4 $H = 0$, since 3–3 is an axis of symmetry for one leg and 4–4 is an axis of symmetry for the other. Applying equation (3),

$$0 = H_0 + 10 \times 1.5 \times 1; \quad H_0 = -15 \text{ inches.}^4$$

6. Find the product of inertia of an 8-inch by 6-inch by 1-inch angle section with respect to axes through the center of gravity parallel to the legs, the section being in a position similar to Fig. 246.

7. A semicircular area of radius a is in the position shown in Fig. 247. Find the product of inertia with respect to the X and Y axes.

$$Ans.\ H = \frac{2a^4}{3}.$$

The product of inertia has no physical significance, but is a convenient mathematical tool in finding the moment of inertia of a plane area with respect to any axis, as will be seen in the article which follows.

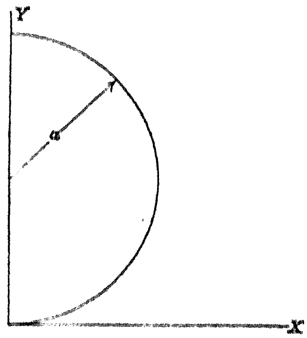


FIG. 247.—Product of inertia of semicircle.

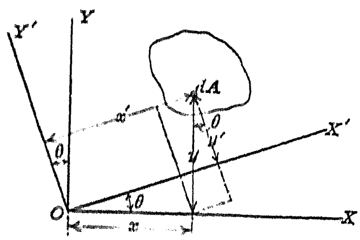


FIG. 248.—Change of direction of axis.

208. Change of Direction of Axis.—Formula XXXIV enables us to transfer moment of inertia from one axis to a parallel axis. It is frequently necessary to transform to an axis at an angle with the original axis.

Fig. 248 represents an area in the XY plane. The moment of inertia of this area with respect to the X axis OX we will call I_x, and the moment of inertia with respect to the Y axis we will call I_y:

$$I_x = \int y^2 dA;$$
$$I_y = \int x^2 dA.$$

Let OX', OY' be new axes making an angle θ with the X and Y axes respectively. The coördinates of the element of area dA with respect to these new axes are (x', y').

The moment of inertia of the area with respect to OX' is

$$I = \int y'^2 dA. \tag{1}$$

From the geometry of the figure

$$y' = y \cos\theta - x \sin\theta. \tag{2}$$

$$I = \int (y^2 \cos^2\theta - 2\,xy \cos\theta \sin\theta + x^2 \sin^2\theta) dA, \tag{3}$$

$$I = I_x \cos^2\theta + I_y \sin^2\theta - 2\cos\theta\sin\theta\int xy\,dA, \tag{4}$$

$$I = I_x \cos^2\theta + I_y \sin^2\theta - H\sin 2\theta. \tag{5}$$

$$I = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - H\sin 2\theta. \tag{6}$$

Problems

1. Find the moment of inertia of a 4-inch by 3-inch rectangle (Fig. 249) with respect to an axis through the lower left corner making an angle of 20 degrees with the 4-inch edge.

$I_x = 36$ inches,[4] $I_y = 64$ inches,[4] $H = 36$ inches.[4]

$$I = \frac{36 + 64}{2} + \frac{36 - 64}{2}\cos 40° - 36\sin 40°.$$

$$I = 50 - 14 \times 0.7660 - 36 \times 0.6428 = 16.14 \text{ inches.}^4$$

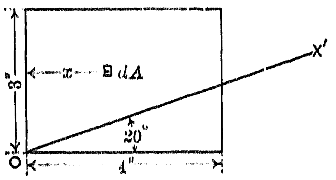


FIG. 249.—Moment of inertia with respect to OX'.

2. Solve Problem 1 if the axis is 20° below direction of the 4-inch edge. *Ans.* $I = 62.42$ inches.[4]

3. Find the moment of inertia of a 3-inch by 4-inch rectangle with respect to a diagonal by means of equation (5) and check by the moment of inertia of two triangles with respect to the diagonal as a common base.

4. Find the moment of inertia of a 4-inch by 4-inch by ½-inch angle section of Fig. 250 with respect to axis 3–3, through center of gravity of section. Take I_x from the table in the handbook. *Ans.* $I = 2.29$ inches.[4]

209. Transformation of Direction of Axes for Product of Inertia.—To get the product of inertia for the axes OX', OY' of Fig. 248, we have:

$$H' = \int x'y'dA. \tag{1}$$

$$H' = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)dA, \tag{2}$$

$$H' = (\cos^2\theta - \sin^2\theta)\int xy\,dA + \cos\theta\sin\theta\int(y^2 - x^2)dA, \tag{3}$$

$$H' = H\cos 2\theta + \frac{I_x - I_y}{2}\sin 2\theta. \tag{4}$$

H' becomes zero when the right member of (4) = 0, when

$$\tan 2\theta = \frac{2H}{I_y - I_x}. \tag{5}$$

Problems

1. In the 4-inch by 3-inch rectangle of Fig. 249 what will be the angle between OX' and the 4-inch edge if the product of inertia with respect to OX' and the axis through O normal to it is zero? *Ans.* $\theta = 34° 22'$.

2. Find the direction of the pair of axes through the center of gravity of the 6-inch by 5-inch by 1-inch angle section of Fig. 246 for which the product of inertia is zero.

210. Direction of Axis for Maximum Moment of Inertia.

Equation (6) of Article 208 is:

$$I = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\,\theta - H \sin 2\,\theta. \qquad (1)$$

Differentiating with respect to θ,

$$\frac{dI}{d\theta} = (I_y - I_x) \sin 2\,\theta - 2\,H \cos 2\,\theta, \qquad (2)$$

from which the condition of maximum or minimum is

$$\tan 2\,\theta = \frac{2\,H}{I_y - I_x}. \qquad (3)$$

Comparing (3) with (5) of Article 209, we find that the condition of maximum or minimum moment of inertia is the condition which gives zero product of inertia. There are two solutions for (3), which give values of $2\,\theta$ differing by 180 degrees and values of θ differing by 90 degrees. One of these positions is that of maximum moment of inertia and the other is that of minimum moment of inertia. Since the product of inertia for an axis of symmetry is zero, the moment of inertia with respect to an axis of symmetry is either greater or less than the moment of inertia for any other axis through any given point in its line.

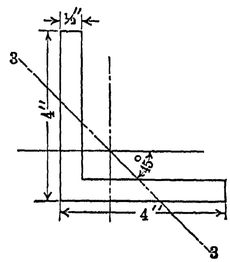


FIG. 250.—Axis of minimum moment of inertia.

The line which bisects the angle between the legs of an angle section of equal legs is a line of symmetry and the moment of inertia for this axis is greater than that for any other axis through the center of gravity, while the moment of inertia for the axis at right angles to this line of symmetry (the axis 3–3 of Fig. 250) is less than that for any other axis through the center of gravity.

Problems

1. For the rectangle of Problem 1 of Article 208 find the maximum and minimum moment of inertia for axes through one corner.

2. Find the maximum and minimum moment of inertia of a 6-inch by 5-inch by 1-inch angle section for axes through the center of gravity. Com-

pare the results for the figures in the handbook for a 3-inch by $2\frac{1}{2}$-inch by $\frac{1}{2}$-inch angle section.

The maximum and minimum moments of inertia of an area for axes through a given point are called the *principal moments of inertia*, and the corresponding axes are the principal axes.

If the minimum moment of inertia is known, it is generally easy to find the maximum by means of a simple relation

$$I_{\max} + I_{\min} = I_x + I_y = J. \tag{4}$$

The sum of the moments of inertia of a plane area for any pair of rectangular axes in the plane is equal to the polar moment of inertia for their point of intersection.

Let one of these axes be used as the X axis.

$$I_x = \int y^2 dA. \tag{5}$$

If the other rectangular axis is used as the Y axis,

$$I_y = \int x^2 dA. \tag{6}$$

For the polar moment of inertia $r^2 = x^2 + y^2$,

$$J = \int (x^2 + y^2) dA = \int x^2 dA + \int y^2 dA. \tag{7}$$

Problems

3. Find the maximum moment of inertia of a 5-inch by 3-inch by $\frac{1}{2}$-inch angle section for an axis through the center of gravity. Get I_x and I_y from the handbook, and calculate min I from the area and the least radius of gyration. Solve for max I by means of equation (4).

4. Find the least and greatest moment of inertia and radius of gyration of a semicircular area of radius a with respect to axes in its plane passing through the end of the diameter which bounds it.

211. Moment of Inertia of a Prism or Pyramid.—The moment of inertia of any solid may be found by triple integration with an element which is infinitesimal in three directions, or by double integration with an element which is infinitesimal in two directions and extends entirely through the mass in the direction of the axis.

It is often easier to use a thin plate or disk which is infinitesimal in one direction only as the element of volume, provided the moment of inertia of this element is known with respect to an axis through its center of gravity parallel to the axis of inertia.

Problems

1. Find the moment of inertia of a right pyramid of height h, with a square base of side b, with respect to an axis through the vertex perpendicular to the base.

The element of volume is the square plate of thickness dx. Its volume is Adx, where A is the area of the section. From similar solids (Fig. 251),

$$A = \frac{b^2x^2}{h^2}.$$

As each side is $\frac{bx}{h}$, its polar moment of inertia with respect to the X axis is $\frac{\rho b^4 x^4}{6\,h^4}dx$. The total moment of inertia is the sum of that of the several plates.

$$I = \frac{\rho b^4 x^5}{30\,h^4} = \frac{\rho b^4 h}{30} = \frac{M b^2}{30}.$$

2. Find the moment of inertia of a right pyramid, the base of which is a hexagon of side a, with respect to an axis through the vertex perpendicular to the base.

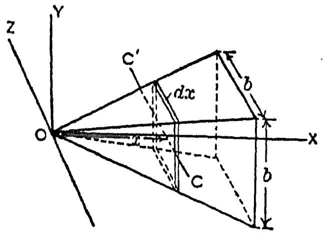


FIG. 251.—Moment of inertia of pyramid.

If in Fig. 251 we wished to find the moment of inertia with respect to the Z axis OZ, we could find the moment of inertia of the plate with respect to the parallel axis CC' and then transfer to the Z axis. The moment of inertia of the plate is the same as that of the area of the plate with respect to a line in its plane multiplied by the thickness dx and the density. The moment of inertia of the plate about CC' is

$$I_0 = \frac{\rho b^4 x^4}{12\,h^4}\,dx;$$

$$M d^2 = \frac{\rho b^2 x^4}{h^2}\,dx,$$

where I_0 and Md^2 have the meaning of Formula XXXIV.

$$I = \rho\left(\frac{b^4 x^5}{60\,h^4} + \frac{b^2 x^5}{5\,h^2}\right) = \frac{\rho b^2 h}{5}\left(\frac{b^2}{12} + h^2\right).$$

Problems

3. Find the square of the radius of gyration of a right pyramid 24 inches high with base 12 inches square with reference to an axis through the vertex parallel to the base. *Ans.* $k^2 = 352.8$ inches.²

4. Find the moment of inertia of a right cylinder of radius a and length l with respect to an axis perpendicular to the axis of the cylinder through the center of one end. *Ans.* $k^2 = \frac{l^2}{3} + \frac{a^2}{4}.$

Observing the answer of Problem 4, we see that the square of the radius of gyration is made of two terms, the first of which is k^2 for a long thin rod with respect to an axis through one end perpendicular to its length, and the other is k^2 for a circular area with respect to a diameter. The moment of inertia of any solid

with a constant cross-section and ending with parallel planes normal to its length (any right prism or cylinder) may be calculated in the same way. Expressed algebraically,

$$k^2 = k_l^2 + k_A^2, \tag{1}$$

where k_l is the radius of gyration of the prism regarded as a thin rod and k_A is the radius of gyration of a cross-section. Fig. 252 represents a triangular prism with its axis parallel to the X axis. It is desired to find its moment of inertia with respect to the Z axis.

Using the element BB' (Fig. 252) of cross-section $dxdy$ extending entirely through the body in the direction of the Z axis:

$$I = \rho \int\int (x^2 + y^2) dx dA \tag{2}$$

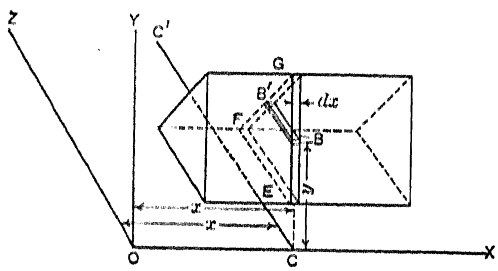


FIG. 252.—Moment of inertia of prism.

where dA is an area of length BB' and height dy.

$$I = \rho \int \int x^2\, dx\, dA + \rho \int \int y^2 dx\, dA \tag{3}$$

When we integrate with respect to dA, x remains unchanged, as we simply pile up elements of the form of BB' from the bottom of the top of the section EFG between vertical planes at a distance dx apart:

$$I = \rho \int x^2 A\, dx + \rho \int I_A dx \tag{4}$$

where A is the area of the section, and I_A is the moment of inertia of the plane area with respect to the axis CC' in the XZ plane parallel to the Z axis.

Equation (4) applies to a solid of any form whatever, and is not limited to a prism as shown in the figure. If the line OX passed through the center of gravity of all the sections, we would have an example of Formula XXXIV as in Problems 3 and 4.

If the solid is a prism or cylinder with the axis parallel to the X axis, A is constant and I_A is constant; then

$$I = \rho A \int x^2\,dx + \rho I_A \int dx \tag{5}$$

The first term of the last member of (5) is the moment of inertia of a thin rod with respect to an axis perpendicular to its length. The second member is equal to $\rho lAk_A^2 = Mk_A^2$, which proves equation (1).

Problems

5. Find the moment of inertia of a right cylinder 18 inches long and 12 inches in diameter with respect to an axis in the plane of one end and tangent to the cylinder. *Ans.* $I = 153\,M$.

6. Find the moment of inertia of a prism 6 inches square and 24 inches long with respect to an axis in the plane of one end perpendicular to the end of a diagonal. *Ans.* $I = M\,(192 + 21)$.

212. Moment of Inertia by Experiment.—A common method of finding the moment of inertia of an irregular body is that of deter-

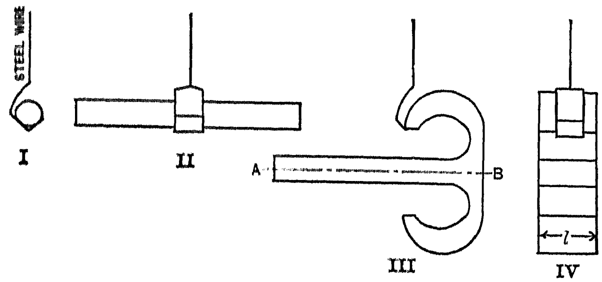


FIG. 253.—Moment of inertia by torsional vibration.

mining its effect upon the time of vibration of a torsion pendulum. The time of vibration of a torsion pendulum varies as the square root of the moment of inertia of its mass with respect to the vertical line which is the axis of the supporting wire. This relation may be expressed briefly:

$$T^2 = KI,$$

where T may be the time of a complete period or of a single vibration (with K varying accordingly), and K is a constant which depends upon the length, diameter, and modulus of shearing elasticity of the supporting wire. The factors which make up K need not be determined separately, as the entire term may be obtained by substitution from the time of vibration of a mass of known moment of inertia.

Fig. 253, I and II, shows a uniform solid circular bronze or brass

rod in a horizontal position on a light support suspended by a single steel wire. Fig. 253, III and IV, shows a second body on the same support. It is desired to find the moment of inertia of this second body with respect to an axis through its center of gravity perpendicular to the line AB. If the body can be so supported that AB is horizontal, it will rotate about the desired axis, for the center of gravity of the combined body and support must fall directly under the axis of the wire, and if the support is relatively small this combined center of gravity will practically coincide with that of the body, even if the support does not hang in exactly the position which it occupies when it is not loaded.

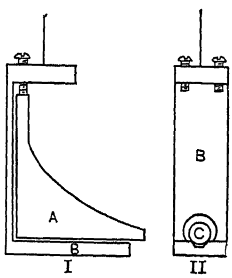


FIG. 254.—Support for torsional vibration.

When the moment of inertia of the support is small, as in Fig. 253, the unknown moment of inertia is calculated from

$$\frac{I_A}{I_C} = \frac{T_A^2}{T_C^2},$$

where the subscript A refers to the body and the subscript C to the cylinder.

Generally it is not practicable to use a very light support and get the body in the desired position. Fig. 254 shows a relatively large support carrying the unknown body in the side elevation of Fig. 254, I, and the known cylinder in the other. In this case we get the time of vibration with the support alone; and then with the support and each load separately.

$$T_B^2 = KI_B,$$
$$T_C^2 = K(I_C + I_B),$$
$$T_A^2 = K(I_A + I_B),$$

where T_C is the time with support and cylinder; T_B, with the support alone, etc.

$$I_A = \frac{(T_A^2 - T_B^2)I_C}{T_C^2 - T_B^2}.$$

Problems

1. The time of vibration of a given torsion pendulum with the support alone is 0.46 second; with the support loaded with a cylinder 10 inches long and ½ inch in diameter it is 0.87 second; with an unknown body in place of the cylinder it is 0.94 second. The cylinder weighs 0.556 pound and the body 1.25 pounds. Find the moment of inertia and radius of gyration of the body. *Ans.* $k = 2.14$ inches.

2. Under what conditions may the unknown moment of inertia be accurately determined without getting the time of vibration of the support?

3. If any clamp screws are used in the support, they should be vertical. Why?

213. The Moment of Inertia of a Plane Section.—The method of the preceding article affords a method of obtaining the moment of inertia of any plane section when the material can be cut up into pieces. Suppose we have a beam of any irregular section. Cut out a piece of some convenient length and finish the ends to parallel planes perpendicular to the length of the beam. A convenient length for the finished piece is 1 inch. Determine the area of cross-section by calculation from the weight and the specific gravity. Get the center of gravity by the method of Article 202. Suspend, and compute the moment of inertia. Divide by the weight for k^2. This k^2 is the square of the radius of gyration of the solid prism 1 inch long.

From equation (1) of Article 211 we know that the square of the radius of gyration of a prism is equal to the sum of the squares of the radius of gyration as a thin section and the radius of gyration as a rod. In this case the square of the radius of gyration as a rod is one-twelfth of the square of the length.

Problems

1. In the case of the unknown section of Fig. 253, III and IV, the length l is 1 inch, the weight in air 1.524 pounds, the weight in water 1.326 pounds. The water was at the temperature at which the density is 62.2 pounds per cubic foot. What is the area of cross-section? *Ans.* 5.50 square inches.

2. On a torsion pendulum with a light support the body in the position shown made 100 vibrations in 83.2 seconds. A rod ½ inch in diameter and 12 inches long weighing 0.668 pound makes 100 vibrations in 163.8 seconds. What is the radius of gyration of the body and of its cross-section?

Ans. k of cross-section is 1.127 inches.

3. In Problem 2 what is the moment of inertia of the cross-section?

Ans. 6.99 inches.^{4}

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